Hamiltonian Theory: Generalizations to Higher Dimensions, Supersymmetry, and Modified Gravity 85

哈密顿理论: 高维推广、超对称与修正引力 85

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Loop quantum gravity in its Hamiltonian form relies on a connection formulation of the gravitational phase space with three key properties: (1.) a compact gauge group, (2.) real variables, and (3.) canonical Poisson brackets. In conjunction, these properties allow to construct a well-defined kinematical quantization of the holonomy-flux algebra, on top of which the remaining constraints can be implemented. While this idea has traditionally been mainly used for Einstein gravity, any gravitational theory with the above properties can be accommodated. In this chapter, we are going to review three strands of work building on this observation, namely, the study of higher-dimensional loop quantum gravity, supersymmetric extensions of loop quantum gravity, as well as the quantization of modified gravitational theories.

哈密顿形式的圈量子引力依赖于引力相空间的联络表述，该表述具备三个核心性质:(1) 紧致规范群；(2) 实变量；(3) 正则泊松括号。这三个性质结合起来，使得我们可以对全环绕-流代数构造定义良好的动力学量子化，在此基础上才能引入其余约束条件。传统上这一思路主要用于爱因斯坦引力，但任何具备上述性质的引力理论都可以适用。本章我们将回顾基于这一结论的三项研究方向，即高维圈量子引力、圈量子引力的超对称扩展，以及修正引力理论的量子化。

Keywords

关键词

Hamiltonian general relativity - Connection dynamics - Supergravity - Supersymmetry - Modified gravity

哈密顿广义相对论 - 联络动力学 - 超引力 - 超对称 - 修正引力

Introductory Remarks

引言

Hamiltonian loop quantum gravity relies at its core on a kinematical construction leading to the Ashtekar-Lewandowski Hilbert space [1-6]. This Hilbert space carries a representation of the classical holonomy-flux algebra that provides a point-splitting subset of phase space functions, thus leading to a kinematical quantization. At this point, there are still constraints to be dealt with via Dirac quantization in order to compute the dynamics. A crucial input at the classical level is a formulation of general relativity (GR) in terms of connection variables with certain properties discussed below. In the seminal papers, mainly the gravitational degrees of freedom were discussed, but matter field can be included in analogy to ideas from lattice gauge theory [7, 8] .

哈密顿圈量子引力的核心是运动学构造，其导出了阿西卡-勒万多夫斯基希尔伯特空间 [1-6]。该希尔伯特空间承载了经典完整流-通量代数的表示，这个代数给出了相空间函数的点分裂子集，进而得到运动学量子化。目前仍有若干约束需要通过狄拉克量子化处理，才能计算动力学。经典层面的关键输入是用联络变量表述的广义相对论 (GR)，我们会在下文讨论该表述的相关性质。在开创性研究中，学者主要讨论了引力自由度，但物质场可以仿照格点规范理论的思路纳入其中 [7, 8] 。

While treating general relativity in 3+1 dimensions along with rather arbitrary matter fields is already a quite satisfactory situation, one may ask to which extent one can apply the techniques developed in loop quantum gravity to other gravitational theories. Motivation to do so can be given as follows:

尽管对 3+1 维广义相对论以及任意物质场的处理已经相当完善，但我们仍然可以探究: 圈量子引力发展出的技术能在多大程度上应用于其他引力理论。开展这一研究的动机如下:

1. Higher-dimensional gravitational theories are mainly interesting from the point of view of making contact with string theory (see, e.g., [9, 10]), which requires 10, 11, or 26 dimensions, respectively, for consistent formulations, as well as applications of the gauge/gravity duality (see, e.g., [11]), where one would, e.g., like to consider loop quantum gravity in 4+1-dimensional anti-de Sitter spacetimes to make statements about 3+1-dimensional gauge theories (see, e.g., [12,13]).

1. 高维引力理论的研究价值主要体现在两个方面: 一是建立与弦理论的联系 (例如见 [9, 10])，弦理论的自洽 formulation 分别要求 10 维、11 维或 26 维时空；二是规范/引力对偶的应用 (例如见 [11])，比如研究者希望通过研究 4+1 维反德西特时空的圈量子引力，得到 3+1 维规范理论的相关结论 (例如见 [12,13])。

2. Historically, supersymmetric gravitational theories have been considered due to their enhanced UV behavior as well as for unification purposes; see, e.g., [14]. Nowadays, they are mainly relevant for string theory and holography, where they appear as low-energy theories and the relevant gravitational theories, respectively; see, e.g., [15]. Next to contact with string theory and holography as above, it is interesting to consider supersymmetric gravitational theories by themselves and to investigate what can be learned about the quantum dynamics due to the more complicated algebra of constraints [16].

2. 从研究历史来看，研究者最初考虑超对称引力理论，是因为它的紫外行为更好，还能实现统一；例如见 [14]。如今，超对称引力理论主要在弦理论和全息学中发挥作用: 它分别作为弦理论的低能理论、全息学的相关引力理论出现；例如见 [15]。除了能和上述弦理论、全息学建立联系之外，超对称引力理论本身就值得研究——由于它的约束代数更复杂，我们可以借此探究量子动力学能得到哪些结论 [16]。

3. Modified gravity theories play an important role in cosmology [17]. Observations strongly implied that our universe is currently undergoing a period of accelerated expansion and is usually referred to as the ”dark energy” issue. The origin of current cosmic acceleration is one of the biggest challenges to modern physics. This issue is hard to be accounted in general relativity (GR) framework. Hence, it is reasonable to consider the possibility that GR is not a valid gravity theory at the cosmological scale and should be modified. If nature is indeed described by such a modified gravitational theory, a loop quantum gravity-type quantization of such theories needs to be investigated as well.

3. 修正引力理论在宇宙学中发挥着重要作用 [17]。观测结果强烈表明，我们的宇宙目前正处于加速膨胀阶段，这就是通常所说的“暗能量”问题。当前宇宙加速膨胀的起源是现代物理学最大的挑战之一，广义相对论 (GR) 框架很难解释这个问题。因此，我们有理由考虑一种可能性: 广义相对论在宇宙学尺度上并非成立的引力理论，需要对其进行修正。如果自然确实由这类修正引力理论描述，那么我们同样需要研究这类理论的圈量子引力型量子化。

As mentioned in the abstract, the three key properties that a connection formulation of a gravitational theory should enjoy in order to directly apply the quantization methods of loop quantum gravity are (1.) a compact gauge group, (2.) real variables, and (3.) canonical Poisson brackets. From the canonical brackets, one obtains the standard form of the holonomy-flux algebra, in particular the commutativity of holonomies among themselves and the action of fluxes as ”grasping operators” which split holonomies at the intersection points. On this algebra, the Ashtekar-Lewandowski functional is positive, linear, and normalized, thus providing a Hilbert space representation. Compactness of the gauge group allows to implement the continuum limit by accounting for cylindrical consistency as well as a rigorous quantization of the constraints. Moreover, the Hilbert space representation yields a useful basis in terms of spin network states. Finally, real variables ensure that the reality conditions are trivially implemented.

正如摘要所述，要直接应用圈量子引力的量子化方法，引力理论的联络表述需要满足三个核心性质:(1) 紧致规范群，(2) 实变量，(3) 正则泊松括号。从正则括号我们可以得到标准形式的完整流-通量代数，尤其是完整流之间的对易性，以及通量作为“抓取算子”在交点处分裂完整流的作用。在这个代数上，阿西卡-勒万多夫斯基泛函是正定、线性且归一化的，因此可以给出希尔伯特空间表示。规范群的紧致性让我们可以通过柱一致性实现连续极限，也能对约束进行严格量子化。此外，该希尔伯特空间表示可以给出自旋网络态形式的有用基。最后，实变量保证实条件可以平凡实现。

However, as we will see explicitly in section ”Quantization of the SUSY Constraint,” in the context of supergravity, it turns out that using real variables leads to a rather complicated form of the constraint operators, in particular the so-called SUSY constraint operator, making direct physical applications and predictions almost impossible. This changes drastically if complex variables are used instead, which recover some of the underlying supersymmetry of the theory and thus simplify the constraints. These observations allow for direct physical applications such as in the context of cosmology and black holes. The prize to pay, however, is the non-compactness of the resulting gauge group and the question on how to implement reality conditions making quantization arguments more subtle.

然而，正如我们会在“超对称约束的量子化”一节明确展示的，在超引力框架下，使用实变量最终会让约束算符 (尤其是所谓的超对称约束算符) 的形式变得十分复杂，导致几乎无法开展直接的物理应用和做出预言。如果改用复变量，情况会发生极大改观: 复变量可以还原出理论部分基础超对称性，从而简化约束。这些结论使得复变量可以直接应用于宇宙学、黑洞等物理场景。不过，需要付出的代价是得到的规范群非紧致，且如何实现实条件这一问题会让量子化的论证变得更微妙。

Higher Dimensions

高维

In this part of the chapter, we review the papers [18-25], where the higher-dimensional connection formulation is developed and the existing quantization techniques have been extended where necessary. Some aspects of boundaries that have been discussed in [26-28] are also mentioned. We omit more recent work on, e.g., coherent states [29], the simplicity constraint [30], and polytopes [31] for brevity.

在本章的这一部分，我们综述文献 [18-25]，这些工作建立了高维联络表述，并在必要处推广了现有量子化技术。我们也会提及 [26-28] 中讨论过的边界相关的部分内容。为简洁起见，我们略去了更近的研究，例如相干态 [29]、简洁性约束 [30] 与多面体 [31] 相关工作。

In short, it was possible to extend the quantization methods of loop quantum gravity to higher dimensions. The key result was a connection formulation of higher-dimensional general relativity with the properties discussed in the introduction [19, 20]. Based on this, the quantization procedure could be straightforwardly adapted, geometric operators for (spatial codimension one) area and volume could be defined, and a quantization of the Hamiltonian constraint could be constructed [21]. Moreover, extensions for most matter fields, including those of several interesting supergravity theories, could be constructed [22, 24, 25]. A key new ingredient, the simplicity constraint, was discussed in [23].

简言之，圈量子引力的量子化方法可以推广到高维。核心结果是得到了具备引言 [19,20] 中所讨论性质的高维广义相对论联络表述。在此基础上，量子化过程可以直接适配，能够定义 (空间余维 1 的) 面积和体积几何算子，还可以构造哈密顿约束的量子化 [21]。此外，可以构造大多数物质场的推广，包括若干有意思的超引力理论的物质场 [22, 24, 25]。文献 [23] 讨论了一个核心新要素: 简洁性约束。

Connection Formulation of Higher-Dimensional GR

高维广义相对论的联络表述

General Considerations

一般考量

As mentioned in the introduction, we want to quantize a Hamiltonian formulation of GR in terms of a connection and its conjugate momentum. As a starting point for the derivation, we use the Arnowitt-Deser-Misner (ADM) formulation of GR [32] in terms of metric variables. As usual, we assume that our $(D+1)$ -dimensional spacetime manifold $\mathcal{M}$ foliates as $\mathbb{R}\times\sum$, where $\sum$ is a D -dimensional Cauchy surface. On $\sum$, the spacetime metric $g_{\mu\nu}, \mu, \upsilon = 0, \ldots, D$ induces a Riemannian metric $q_{ab}, a, b = 1, \ldots, D$. The extrinsic curvature of $\sum$ in $\mathcal{M}$ is denoted by K_{ab} . We use the convention

如引言所述，我们希望用联络及其共轭动量对广义相对论的哈密顿表述进行量子化。作为推导的起点，我们采用度量变量下的阿尔诺维特-德泽-米斯纳 (ADM) 广义相对论表述 [32]。和通常一样，我们假设我们的 $(D+1)$ 维时空流形 $\mathcal{M}$ 分叶为 $\mathbb{R}\times\sum$ ，其中 $\sum$ 是 D 维柯西面。在 $\sum$ 上，时空度量 $g_{\mu\nu}, \mu, \upsilon = 0, \ldots, D$ 诱导出一个黎曼度量 $q_{ab}, a, b = 1, \ldots, D$ 。$\sum$ 在 $\mathcal{M}$ 中的外曲率记为 K_{ab} 。我们采用的约定是

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int_{\mathcal{M}}\sqrt{-g}R\left(g\right)d^{D+1}x \tag{1}$$

for the Einstein-Hilbert action, which leads to the ADM Poisson bracket

针对爱因斯坦-希尔伯特作用量，由此可得到 ADM 泊松括号

$$\left\{q_{ab}\left(x\right), P^{cd}\left(y\right)\right\} = \delta^{(D)}\left(x-y\right)\delta^{c}_{(a}\delta^{d}_{b)} \tag{2}$$

with $P^{ab} = \frac{1}{2\kappa}\sqrt{q}\left(K^{ab} - q^{ab}K^{cd}q_{cd}\right)$. We will set $\kappa = 8\pi G = 1$ in the following. The Hamiltonian of the theory is given by a sum of the Hamiltonian constraint $S\left(x\right)$ and the spatial diffeomorphism constraint $V_a\left(x\right)$, smeared against the lapse function $N\left(x\right)$ and shift vector $N^a\left(x\right)$, giving

满足 $P^{ab} = \frac{1}{2\kappa}\sqrt{q}\left(K^{ab} - q^{ab}K^{cd}q_{cd}\right)$ 。下文中我们令 $\kappa = 8\pi G = 1$ 。该理论的哈密顿量由哈密顿约束 $S\left(x\right)$ 和空间微分同胚约束 $V_a\left(x\right)$ 分别对时移函数 $N\left(x\right)$ 和位移矢量 $N^a\left(x\right)$ 做弥散求和得到，即

$$H_{\mathrm{ADM}} = \int_{\sum}\left(N\left(x\right)S\left(x\right) + N^a\left(x\right)V_a\left(x\right)\right)d^Dx. \tag{3}$$

For later reference, we explicitly state

为便于后续参考，我们明确写出

$$S = -\frac{s}{\sqrt{\det\left(q\right)}}\left[q_{ac}q_{bd} - \frac{1}{D-1}q_{ab}q_{cd}\right]P^{ab}P^{cd} - \sqrt{\det\left(q\right)}R^{(D)}, \tag{4}$$

where s represents the signature of spacetime as -1 for Lorentzian and +1 for Euclidean.

其中 s 表示时空符号号差，洛伦兹号差为-1，欧几里得号差为 +1。

The technical means for obtaining a connection formulation starting from the ADM formulation is that of a phase space extension. The basic idea is to postulate a new, larger phase space, subject to additional first-class constraints. This phase space should be coordinatized by a connection $A_{a\alpha}, \alpha$ being a Lie algebra index in a Lie group $\mathcal{G}$ and its canonically conjugate (We exclude the possibility of having noncanonical brackets between $A_{a\alpha}$ and $\pi^{a\alpha}$, since this would lead to a non-standard holonomy-flux algebra, on which is the Ashtekar-Lewandowski measure might not lead to a positive linear functional.) momentum $\pi^{a\alpha}$. The list of constraints should include a Gauß law $G^{\alpha}(x) := D_a \pi^{a\alpha}(x) = 0$, which generates local $\mathcal{G}$ gauge transformations. Furthermore, we need explicit expressions of the ADM variables q_{ab} and P^{ab} in terms of $A_{a\alpha}$ and $\pi^{a\alpha}$. Then, what one has to proof is that the ADM Poisson brackets are reproduced up to the new constraints, that is,

从 ADM 表述出发得到联络表述的技术手段是相空间延拓。基本思路是引入一个满足额外第一类约束的更大的新相空间，该相空间的坐标取为联络 $A_{a\alpha}, \alpha$ (Lie 群 $\mathcal{G}$ 中的李代数指标) 及其正则共轭动量 $\pi^{a\alpha}$ (我们排除了 $A_{a\alpha}$ 和 $\pi^{a\alpha}$ 之间存在非正则括号的可能，因为这会得到非标准的全纯-流代数，阿西卡-莱万多夫斯基测度在这类代数上可能无法给出正线性泛函)。约束集中应包含高斯定律 $G^{\alpha}(x) := D_a \pi^{a\alpha}(x) = 0$ ，它生成局域 $\mathcal{G}$ 规范变换。此外，我们需要给出 ADM 变量 q_{ab} 、P^{ab} 用 $A_{a\alpha}$ 和 $\pi^{a\alpha}$ 表示的显式表达式。接下来需要证明 ADM 泊松括号在新约束允许的范围内可以被还原，即

$$\left\{q_{ab}[A,\pi](x), P^{cd}[A,\pi](y)\right\} \approx \delta^{(D)}(x-y)\,\delta^{c}_{(a}\delta^{d}_{b)}, \tag{5}$$

where $\approx$ denotes a weak equality, that is, equality up to constraints. The remaining Poisson brackets have to vanish. Finally, one expresses S and V^a in terms of $A_{a\alpha}$ and $\pi^{a\alpha}$. The physics of the new formulation is now equivalent to the ADM formulation, and the extra degrees of freedom constitute a gauge redundancy in the description.

其中 $\approx$ 表示弱等式，也就是差一个约束的等式。其余泊松括号必须为零。最后，我们将 S 和 V^a 用 $A_{a\alpha}$ 和 $\pi^{a\alpha}$ 表示出来。此时新表述的物理内容和 ADM 表述完全等价，额外自由度只是描述引入的规范冗余。

A priori, there is a vast landscape of possible connection formulations, and one would like to select an appropriate one. It turns out when considering only the Gauß law as a new constraint and a vielbein-type construction of the connection variables, the possible gauge groups are restricted to $\mathrm{SO}(3)$ and $\mathrm{SO}(1,2)$ or their respective universal covers [19]. The corresponding canonical variables are then either the drei-bein and its conjugate connection in 2+1 gravity or the Ashtekar-Barbero variables [33,34] in $3+1$ dimensions.

先验地看，可能的联络表述数量非常庞大，我们需要从中选出合适的一个。研究发现，如果仅将高斯定律作为新约束，且采用标架型方式构造联络变量，那么可能的规范群只能是 $\mathrm{SO}(3)$ 和 $\mathrm{SO}(1,2)$ 或是它们各自的泛覆盖 [19]。对应的正则变量要么是 2+1 维引力中的三标架及其共轭联络，要么是 $3+1$ 维中的阿希卡-巴贝罗变量 [33,34]。

While the quantization and solution of the Gauß law are straightforward in loop quantum gravity, we would furthermore like to demand that the quantization of the additional constraints is ”well behaved.” For example, if the additional constraints could be expressed solely in terms of fluxes, that is, $\pi^{a\alpha}$ smeared over a spacetime codimension 2 surface, they would leave the graph on which spin networks are defined invariant. This in turn means that the kernel of these constraints can be constructed at the level of individual spin networks, i.e., by restricting the allowed representations and invariant maps. In other words, it would be a purely group theoretical exercise and not involve (infinite) superpositions of spin networks, which might be non-normalizable in the kinematical scalar product.

虽然在圈量子引力中高斯定律的量子化与求解都十分直接，但我们进一步要求额外约束的量子化“表现良好”。例如，如果额外约束可以仅用流 (也就是弥散在时空余维 2 曲面上的 $\pi^{a\alpha}$) 表示，那么它们就能保持自旋网所定义的图不变。这反过来意味着这些约束的核可以在单个自旋网层面构造，也就是通过限制允许的表示和不变映射来实现。换句话说，这纯粹是一个群论问题，不会涉及可能在运动学标量积中不可归一化的自旋网 (无穷) 叠加。

While we cannot exclude the existence of further connection formulations with the abovementioned properties, the only two known at present are motivated by the Palatini action. Here, the gauge group is either $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1,D)$, starting with Euclidean or Lorentzian spacetime signature. As we will see however later, at the level of the Hamiltonian theory, both formulations can be used for either Euclidean or Lorentzian gravity, since the signature of spacetime is only encoded in a relative sign in the Hamiltonian constraint. This, at first unintuitive property, becomes clear when considering that in both formulations, the ADM variables are a Riemannian D-metric and its conjugate momentum. A phase space extension performed in either the Euclidean or Lorentzian thus has to be valid also in the other theory. For quantization purposes, the compact group $\mathrm{SO}(D+1)$ is strongly preferred by the currently available techniques in loop quantum gravity, and we will choose this formulation.

虽然我们不能排除存在其他满足上述性质的联络表述, 但目前已知的仅有的两种都是由帕拉蒂尼作用量启发得到的。在这里，从欧几里得或洛伦兹号差的时空出发，规范群要么是 $\mathrm{SO}(D+1)$ 要么是 $\mathrm{SO}(1,D)$ 。但我们后文会看到，在哈密顿理论层面，两种表述都既可以用于欧几里得引力，也可以用于洛伦兹引力，因为时空号差仅编码在哈密顿约束的一个相对符号中。这个乍看反直觉的性质其实很清晰: 两种表述中，ADM 变量都是黎曼 D 度量及其共轭动量，因此无论是欧几里得还是洛伦兹理论中做的相空间延拓，必然对另一种理论也成立。就量子化而言，圈量子引力现有技术非常偏好紧致群 $\mathrm{SO}(D+1)$ ，因此我们将选择这个表述。

The Palatini Action

帕拉蒂尼作用量

We start our motivation for the choice of connection variables by performing a canonical $D+1$ decomposition of the Lorentzian Palatini action

我们通过对洛伦兹帕拉蒂尼作用量做标准的 $D+1$ 分解，来引出我们对联络变量的选择

$$S_{\text{Palatini}} = -\frac{1}{2}\int_{\mathcal{M}} e e^{\mu I} e^{\mu J} F_{\mu\nu IJ}(A)\, d^{D+1}x =: \int_{\mathcal{M}} \sum^{IJ}(e) \wedge F_{IJ}(A), \tag{6}$$

where e_I^μ is a $(D+1)$ -bein, $e = \det e_{\mu I}, A_{aIJ}$ an $\mathrm{SO}(1,D)$ connection, and $F_{\mu\nu IJ}$ its curvature. The global sign of the action is chosen to agree with the conventions in [20], where the following calculations are detailed. Assuming $\mathcal{M} = \mathbb{R} \times \sum$, with $\sum$ compact without boundary, we have

其中 e_I^μ 是 $(D+1)$ 标架，$e = \det e_{\mu I}, A_{aIJ}$ 是 $\mathrm{SO}(1,D)$ 联络，$F_{\mu\nu IJ}$ 是该联络的曲率。我们选取作用量的整体符号与文献 [20] 中的约定一致，下述计算的细节也出自该文献。假设 $\mathcal{M} = \mathbb{R} \times \sum$ ，且 $\sum$ 是无边界紧致流形，我们得到

$$S_{\text{Palatini}} = \int dt \int_{\Sigma} d^D x \left(\frac{1}{2} \pi^{aIJ} \mathcal{L}_T A_{aIJ} - N\mathcal{S} - N^a V_a - \frac{1}{2} \lambda_{IJ} G^{IJ} \right), \tag{7}$$

where $\pi^{aIJ} = 2n^{[I} \sqrt{q} e^{a|J|}$, with $n^I = n^\mu e_\mu^I$ being the internal version of the timelike unit normal n^μ to $\sum$ and $\mathcal{L}_T$ denoting the Lie derivative with respect to the time evolution vector field $T^\mu = Nn^\mu + N^\mu.S$ and V_a are again the Hamiltonian and spatial diffeomorphism constraints, and $G^{IJ} = D_a \pi^{aIJ} = \partial_a \pi^{aIJ} + [A, \pi^a]^{IJ}$ is the Gauß constraint with smearing functions λ_{IJ} . Starting from here, one can perform a canonical analysis following Dirac [35]. While this can be done in all detail [20], the problem is that one would have to introduce additional variables, the momenta to A_{aIJ} and e^{aI} . However, we are interested in a formulation in terms of variables A_{aIJ} and π^{aIJ} only. This can be accomplished by the following observation: if we just use A_{aIJ} and π^{aIJ} as conjugate variables, forgetting about the fact that π^{aIJ} decomposes as $2n^{[I} \sqrt{q} e^{a|J|}$, we would make an error, since we would have too many degrees of freedom. However, we also notice that S, V_a , and G^{IJ} can be expressed purely in terms of A_{aIJ} and π^{aIJ} [20], keeping in mind that $\pi^{aIJ} = 2n^{[I} \sqrt{q} e^{a|J|}$. This leads us to the conclusion that if we can write a constraint purely in terms of A_{aIJ} and π^{aIJ} such that $\pi^{aIJ} = 2n^{[I} \sqrt{q} e^{a|J]}$ on the constraint surface, we can use the phase space coordinatized by A_{aIJ} and a "generic" momentum π^{aIJ} , subject to this additional constraint.

其中 $\pi^{aIJ} = 2n^{[I} \sqrt{q} e^{a|J|}$ 里，$n^I = n^\mu e_\mu^I$ 是类时单位法矢 n^μ 到 $\sum$ 的内版本，$\mathcal{L}_T$ 是时间演化矢量场 $T^\mu = Nn^\mu + N^\mu.S$ 对应的李导数，V_a 仍是哈密顿约束和空间微分同胚约束，$G^{IJ} = D_a \pi^{aIJ} = \partial_a \pi^{aIJ} + [A, \pi^a]^{IJ}$ 是带有弥散函数 λ_{IJ} 的高斯约束。从此出发，可以遵循狄拉克 [35] 的方法进行正则分析。虽然这一过程可以得到全部细节 [20]，但问题在于必须引入额外变量，即对应 A_{aIJ} 和 e^{aI} 的动量。然而我们希望得到一个仅由变量 A_{aIJ} 和 π^{aIJ} 表示的表述。我们可以通过下述观察实现这一点: 如果我们直接用 A_{aIJ} 和 π^{aIJ} 作为共轭变量，忽略 π^{aIJ} 可以分解为 $2n^{[I} \sqrt{q} e^{a|J|}$ 的事实，我们会出错，因为这会导致自由度数量过多。不过我们也注意到，若牢记 $\pi^{aIJ} = 2n^{[I} \sqrt{q} e^{a|J|}$ ，则 S, V_a 和 G^{IJ} 都可以仅用 A_{aIJ} 和 π^{aIJ} 表示 [20]。由此我们得到结论: 如果我们可以写出一个仅由 A_{aIJ} 和 π^{aIJ} 表示的约束，使得在约束曲面上满足 $\pi^{aIJ} = 2n^{[I} \sqrt{q} e^{a|J]}$ ，我们就可以使用由 A_{aIJ} 和“广义”动量 π^{aIJ} 坐标化的相空间，仅需额外满足这个约束。

Surprisingly, it is possible to write down such a constraint in the simple form

令人惊讶的是，我们可以将这个约束写成非常简单的形式

$$S^{abIJKL} := \pi^{a[IJ} \pi^{b|KL]} = 0, \tag{8}$$

which is closely related to the Plebanski formulation of GR [36]. In loop quantum gravity, this constraint is known as the (quadratic (One can also write down a linear version of this constraint; see, for example, [24,37]. The quadratic constraint in 3+1 dimensions suffers from the problem of admitting a topological sector, which has to be excluded by hand. See also [38] for further discussion on related subtleties.)) simplicity constraint and first appeared in the Barrett-Crane spin foam model [39]. It satisfies the above requirement of being expressible in terms of fluxes only and indeed translates into a restriction of the allowed group representation labels in the quantum theory, as we will discuss later.

它与广义相对论的普莱班斯基表述 [36] 密切相关。在圈量子引力中，该约束被称为 (二次型 (也可写出该约束的线性版本，参见例如 [24,37]。3+1 维下的二次型约束存在允许拓扑 sector 的问题，必须手动排除。相关细节的进一步讨论参见 [38]。) 简单性约束，最早出现在巴雷特-克兰自旋泡沫模型 [39] 中。它满足仅能用通量表示的上述要求，在量子理论中确实转化为对允许群表示标签的限制，我们后文会对此展开讨论。

With this constraint added to the list of constraints of the theory, we now perform a canonical analysis. As detailed in [20], this leads to a new constraint which originates from demanding the simplicity constraint to be preserved by the Hamiltonian evolution. We will denote this constraint by D^{abIJKL} , suppressing indices in what follows for notational simplicity. Its precise form doesn't matter for this chapter, and we just remark that it sets a certain part of the torsion of A_{aIJ} to zero. D turns out to be a second-class partner to S , which is unwanted, because it was our aim to impose S^{abIJKL} as a strong operator equation in the quantum theory. As is well known, second-class constraints require different quantization techniques, and D furthermore depends on A_{aIJ} , leading to a complicated operator.

在该理论的约束列表中添加这一约束后，我们现在进行正则分析。正如文献 [20] 中详述，这会产生一个新约束，它源于要求简单性约束在哈密顿演化下保持不变。我们将该约束记为 D^{abIJKL} ，下文为简化记号省略指标。它的具体形式对本章而言无关紧要，我们只需指出，它将 A_{aIJ} 挠率的某一部分置零。结果表明 D 是 S 的二类伙伴，这是我们不想要的，因为我们的目标是在量子理论中将 S^{abIJKL} 作为强算符方程施加。众所周知，二类约束需要不同的量子化技术，此外 D 依赖于 A_{aIJ} ，这会导致算符结构复杂。

A way out of this problem is the technique of gauge unfixing [40-42], which transforms a second-class constrained system in an equivalent first-class constrained system. It can be seen as the Hamiltonian analog of the Stückelberg trick and essentially amounts to the physical equivalence of a theory with gauge freedom before and after gauge fixing. Using this technique, we can view D as a gauge fixing condition for the simplicity constraint and remove it from the list of constraints. We note that S trivially Poisson commutes with itself. The only modification this amounts to is to change the Hamiltonian constraint by essentially adding D^2 in such a way that it Poisson commutes with the simplicity constraint. On the constraint surface of the simplicity and Gauß constraints, the new "gauge unfixed" Hamiltonian constraint then reduces to its ADM version.

解决该问题的一种方法是规范解固定技术 [40-42]，该技术将二类约束系统转化为等价的一类约束系统。它可以被看作是施蒂克尔贝格技巧在哈密顿框架下的对应，本质上等同于规范固定前后带规范自由度的理论在物理上等价。利用该技术，我们可以将 D 看作简单性约束的规范固定条件，并将其从约束列表中移除。我们注意到 S 泊松对易于自身是平凡的。该方法带来的唯一修改是改变哈密顿约束，本质上是添加 D^2 ，使得哈密顿约束与简单性约束泊松对易。在简单性约束和高斯约束的约束面上，新的“经规范解固定的”哈密顿约束就约化为其 ADM 形式。

We thus conclude that we found a Hamiltonian formulation of GR with the following properties:

因此我们得出结论，我们得到了广义相对论的哈密顿表述，它具备以下性质:

- The phase space is of Yang-Mills type, coordinatized by the $\mathrm{SO}(1, D)$ connection A_{aIJ} and its conjugate momentum π^{aIJ} .

- 相空间为杨-米尔斯型，由 $SO(1,D)$ 联络 A_{aIJ} 及其共轭动量 π^{aIJ} 坐标化。

- The Poisson brackets are canonical, i.e., $\left\{A_{aIJ}(x), \pi^{bKL}(y)\right\} = 2\delta_a^b \delta_{[I}^K \delta_{J]}^L \delta^{(D)}(x-y)$.

- 泊松括号是正则的，即 $\left\{A_{aIJ}(x), \pi^{bKL}(y)\right\} = 2\delta_a^b \delta_{[I}^K \delta_{J]}^L \delta^{(D)}(x-y)$。

- The canonical variables are real. - The theory is subject to the first-class constraints $\mathcal{S}, V_a, G^{IJ}$, and S^{abIJKL} .

- 正则变量为实变量。- 理论受一类约束 $\mathcal{S}, V_a, G^{IJ}$ 和 S^{abIJKL} 约束。

- The Hamiltonian is a sum of the (smeared) first-class constraints.

- 哈密顿量是 (弥散后的) 一类约束之和。

Up to the non-compact gauge group, we have thus found a suitable Hamiltonian connection formulation of GR in any dimension $D+1 \geq 3$. In $2+1$ dimension, the simplicity constraints are trivially zero, and the formulation thus simplifies to the usual canonical formulation of the $2+1$ -dimensional Palatini action.

除去非紧致规范群的问题，我们已经得到了任意维度下广义相对论的一个合适的哈密顿联络表述 $D+1 \geq 3$ 。在 $2+1$ 维中，简单性约束平凡为零，因此该表述简化为 $2+1$ 维帕拉蒂尼作用量的常规正则表述。

Compact Internal Group

紧致内群

The only shortcoming of the above canonical formulation of GR which prevents us from applying the quantization techniques on which loop quantum gravity in $3+1$ dimensions is based is the non-compactness of the gauge group, which enters the construction of the Hilbert space by providing a normalizable Haar measure. From an aesthetical point of view, one can argue that having $SO(1,D)$ as an internal gauge group would better reflect the physics of the theory, due to its origin in the Palatini action, whereas its compact analog $SO(D+1)$ results from the Euclidean theory. On the other hand, the internal gauge group is completely redundant for the physics at the classical level, since by solving the simplicity constraint and Gauß constraint, the theory can be reduced to the ADM phase space, which doesn't care from which internal gauge group it has originally been obtained. Its dynamics is solely governed by the remaining constraints, the Hamiltonian and spatial diffeomorphism constraints.

上述广义相对论正则表述唯一阻碍我们应用 $3+1$ 维圈量子引力所基于的量子化技术的缺点是规范群的非紧致性，规范群通过提供可归一化哈尔测度进入希尔伯特空间的构造。从美学角度来看，可以说将 $\mathrm{SO}(1,D)$ 作为内规范群能更好地反映理论的物理性质，这源于其帕拉蒂尼作用量的起源，而其紧致对应物 $\mathrm{SO}(D+1)$ 则来自欧几里得理论。另一方面，内规范群对于经典层面的物理而言是完全冗余的，因为通过求解简约约束和高斯约束，该理论可以约化为 ADM 相空间，ADM 相空间并不关心它最初源自哪个内规范群。其动力学完全由剩余约束——哈密顿约束和空间微分同胚约束——主导。

This now leads us to the following observation: consider the Euclidean Palatini action, and perform a canonical analysis as in the previous section. We obtain the same theory, up to the fact that the internal gauge group now is $\mathrm{SO}(D+1)$ and the Hamiltonian constraint encodes the dynamics of the Euclidean theory. Upon solution of the simplicity and Gauß constraints, we obtain the Euclidean ADM phase space, whose only difference to the Lorentzian ADM phase space is the Hamiltonian constraint, more precisely a relative sign s between the two terms in (4). It now follows that we can write down a theory based on the $\mathrm{SO}(D+1)$ connection variables coming from the Euclidean Palatini action by modifying the Hamiltonian constraint such that it reduces to the Lorentzian ADM constraint upon solving the Gauß and simplicity constraints. This is indeed possible, as detailed in [19].

这由此引出了以下观察: 考虑欧几里得帕拉蒂尼作用量，按照上一节的方法进行正则分析。我们得到了相同的理论，唯一区别在于此时内规范群是 $\mathrm{SO}(D+1)$，且哈密顿约束编码了欧几里得理论的动力学。求解简约约束和高斯约束后，我们得到欧几里得 ADM 相空间，它和洛伦兹 ADM 相空间的唯一区别在于哈密顿约束，更准确地说是 (4) 式两项之间的相对符号 s 。由此我们可以写出一个基于欧几里得帕拉蒂尼作用量导出的 $\mathrm{SO}(D+1)$ 联络变量的理论，方法是修改哈密顿约束，使其在求解高斯约束和简约约束后约化为洛伦兹 ADM 约束。正如文献 [19] 中详细说明的，这确实是可行的。

Let us summarize our findings for now: We have a Hamiltonian theory based on an $\mathrm{SO}(D+1)$ connection and its canonical momentum π^{aIJ}, subject to the canonical brackets

现在我们总结目前的结论: 我们得到了一个基于 $\mathrm{SO}(D+1)$ 联络及其正则动量 π^{aIJ} 的哈密顿理论，它满足正则对易关系

$$\left\{A_{aIJ}(x), \pi^{bKL}(y)\right\} = 2\delta_a^b \delta_{[I}^K \delta_{J]}^L \delta^{(D)}(x-y). \tag{9}$$

The spatial metric q_{ab} is encoded as

空间度规 q_{ab} 可表示为

$$2qq^{ab} = \pi^{aIJ}\pi^b_{IJ}, \tag{10}$$

while the relation to the extrinsic curvature, and thus P^{ab}, is a little more involved. For it, we note that there exists a unique $\mathrm{SO}(D+1)$ "hybrid spin connection" $\Gamma^{\mathrm{H}}_{aIJ}$ with the property $D_a^{\Gamma^{\mathrm{H}}} n^I = D_a^{\Gamma^{\mathrm{H}}} e_I^b = 0$ [43], where $D_a^{\Gamma^{\mathrm{H}}}$ acts on tensor indices with the Christoffel symbols. It can be expressed as a function of π^{aIJ} only [19]. Now, A_{aIJ} decomposes as

而它与外曲率的关系，即 P^{ab} ，要稍微复杂一些。对此，我们注意到存在唯一的 $SO(D+1)$ “混合自旋联络” $\Gamma^{\mathrm{H}}_{aIJ}$ 满足性质 $D^{\Gamma^{\mathrm{H}}}_a n^I = D^{\Gamma^{\mathrm{H}}}_a e^b_I = 0$ [43]，其中 $D^{\Gamma^{\mathrm{H}}}_a$ 作用在带克里斯托费尔符号的张量指标上。它可以表示为仅关于 π^{aIJ} 的函数 [19]。现在，A_{aIJ} 可分解为

$$A_{aIJ} = \Gamma^{\mathrm{H}}_{aIJ}(\pi) + K_{ab}\pi^b_{IJ}/\sqrt{q} + \bar{K}_{aIJ}, \tag{11}$$

where $\bar{K}_{aIJ}$, subject to $\bar{K}_{aIJ}n^I = 0$, is pure gauge [19]. It now can be shown that (5) is satisfied up to the Gauß and simplicity constraints

其中满足 $\bar{K}_{aIJ}n^I = 0$ 条件的 $\bar{K}_{aIJ}$ 是纯规范 [19]。可以证明，(5) 式在高斯约束和简约约束之外成立

$$\begin{aligned} G^{IJ}[\lambda_{IJ}] = \int_{\Sigma} \lambda_{IJ} D_a \pi^{aIJ} d^D x,\ S^{abIJKL}[c_{abIJKL}] \\ := \int_{\Sigma} c_{abIJKL} \pi^{a[IJ} \pi^{b|KL]} d^D x = 0. \end{aligned} \tag{12}$$

c_{abIJKL} has density weight -1 . The spatial diffeomorphism constraint reads (up to a boundary term obtained from the York-Gibbons-Hawking [45] boundary term in the action)

c_{abIJKL} 的密度权重为-1。空间微分同胚约束可写为 (作用量中约克-吉本斯-霍金 [45] 边界项导出的边界项忽略不计)

$$V_a[N^a] = \frac{1}{2}\int_{\Sigma} \pi^{aIJ} \mathcal{L}_N A_{aIJ} d^D x, \tag{13}$$

with $\mathcal{L}_N$ being the Lie derivative with respect to N^a . The form of the Lorentzian Hamiltonian constraint in these variables has been given in [19].

其中 $\mathcal{L}_N$ 是关于 N^a 的李导数。这些变量下洛伦兹哈密顿约束的形式已在文献 [19] 中给出。

Since we already allowed us to use a ”non-standard” formulation to describe GR, we can check whether there is an additional possibility to modify the variables further. Indeed, a simple modification results from rescaling the K_{ab} part in A_{aIJ} and (inversely so) the π^{aIJ} by a non-zero real parameter β (The free parameter β is analogous to the Barbero-Immirzi parameter γ [34,44] known from 3+1 dimensions; however, it is different in the sense that in $3+1$ dimensions, a two-parameter family of connection formulations in terms of β and γ exists [20] and γ is restricted to $3+1$ dimensions only. Moreover, in even spacetime dimensions, one can consider yet another modification of the connection variables related to the θ -ambiguity in QCD; see [104].). Then, the new relation to the ADM phase space is

既然我们已经允许采用“非标准”表述描述广义相对论，我们可以进一步检验是否存在额外可能进一步修改变量。实际上，一个简单的修改是通过非零实参数 β 对 A_{aIJ} 中的 K_{ab} 部分以及对 π^{aIJ} 做缩放 (后者缩放方向相反)。自由参数 β 类似于 3+1 维中已知的巴贝罗-伊米尔齐参数 γ [34,44]；但二者存在区别: 在 $3+1$ 维中，存在由 β 和 γ 构成的两参数联络表述族 [20]，且 γ 仅存在于 $3+1$ 维中。此外，在偶数时空维中，还可以考虑联络变量的另一种修改，该修改和量子色动力学中的 θ 不确定性有关；参见 [104]。那么，它与 ADM 相空间的新关系为

$$\frac{2}{\beta^2} q q^{ab} = {}^{(\beta)}\pi^{aIJ(\beta)}\pi^b_{IJ}, {}^{(\beta)}A_{aIJ} = \Gamma^{\mathrm{H}}_{aIJ}\left({}^{(\beta)}\pi\right) + \beta K_{ab}\left(\beta^{(\beta)}\pi^b_{IJ}\right)/\sqrt{q} + \bar{K}_{aIJ},$$

(14)

with ${}^{(\beta)}\pi^{aIJ} = \pi^{aIJ}/\beta$, while (9) remains unchanged, and in (12) and (13), the rescaled variables simply replace the old ones.

其中满足 ${}^{(\beta)}\pi^{aIJ} = \pi^{aIJ}/\beta$ ，同时 (9) 保持不变，且在 (12) 和 (13) 中，缩放后的变量直接替换原变量即可。

The free parameter β plays an important role in the quantum theory. While classically it amounts to performing a canonical transformation and thus doesn't change the physics, it turns out that these canonical transformations cannot be implemented as unitary transformations in the quantum theory or even algebra automorphisms on the holonomy-flux algebra. We restricted to real β in order to obtain a real connection formulation, whose reality conditions are implemented by the kinematical scalar product. However classically, complex values of β would work equally well.

自由参数 β 在量子理论中发挥着重要作用。从经典层面看，它仅对应一个正则变换，因此不会改变物理内容，但事实证明，这些正则变换无法作为幺正变换在量子理论中实现，甚至无法作为全纯-流代数上的代数自同构实现。我们将范围限制在实 β ，以得到实联络表述，其实数条件由运动学标积实现。但从经典层面看，β 取复数值也同样适用。

Boundaries

边界

Boundaries have been widely investigated within loop quantum gravity; in particular, for the purpose of computing the black hole entropy, see the respective chapters in this book. In this chapter, we include a brief discussion about higher dimensions to show that the results from 3+1 dimensions directly generalize; see [26-28] for details.

边界已在圈量子引力中得到广泛研究；特别是为计算黑洞熵，可参见本书相应章节。本章我们简要讨论高维情形，以展示 3+1 维的结果可以直接推广；细节参见文献 [26-28]。

For the treatment of general boundaries, we will choose a complementary approach to the covariant canonical description and build on existing results in ADM variables [45]. As boundary condition, we choose to fix the induced metric on the boundary, which leads to the well-known York-Gibbons-Hawking boundary term. This boundary term then has to be taken into account in the $(D+1)$ split of the action, as is done in [45]. It results in boundary terms in the spatial diffeomorphism and Hamiltonian constraints. Most relevant for us, the spatial diffeomorphism constraint, including its boundary contribution (We include the boundary term of the constraint in order for it to generate spatial diffeomorphisms also at the boundary. The surface area A_H of a boundary slice is invariant under such spatial diffeomorphisms, and it is thus physically sensible to mod them out also in the quantum theory for the purpose of computing the black hole entropy, which is classically determined by A_H .), is given by

对于一般边界的处理，我们将采用协变正则描述的互补方法，基于 ADM 变量中的现有结果展开研究 [45]。我们选择固定边界上的诱导度量作为边界条件，由此得到著名的约克-吉本斯-霍金边界项。和文献 [45] 中的处理一样，该边界项必须纳入作用量的 $(D+1)$ 分解中。它会在空间微分同胚约束和哈密顿约束中产生边界项。对我们而言最相关的是，包含边界贡献的空间微分同胚约束 (我们纳入该约束的边界项，是为了让它在边界处也能生成空间微分同胚。边界切片的表面积 A_H 在这类空间微分同胚变换下不变，因此在量子理论中扣除这些变换在物理上是合理的——这对于计算由 A_H 经典决定的黑洞熵而言尤为必要。) 由下式给出

$$V_a\left[N^a\right] = \int_\Sigma P^{ab} \mathcal{L}_N q_{ab}, \tag{15}$$

where the shift vector is restricted to satisfy $s_a N^a = 0$ on H in order to preserve the boundary. The symplectic structure has no boundary contribution at the level of the ADM variables.

其中位移矢量需要满足 $s_a N^a = 0$ 在 H 上的条件，以保持边界不变。辛结构在 ADM 变量层面没有边界贡献。

For the phase space extension to connection variables, we need to check whether it leads to a boundary contribution in the symplectic structure and in the constraints when expressed in a suitable form. At the level of the symplectic potential, it can be shown that

将相空间推广到联络变量时，我们需要验证: 当用合适的形式表示后，这是否会在辛结构和约束中产生边界贡献。可以证明，在辛势层面有

$$\int_\Sigma P^{ab} \delta q_{ab} \approx \frac{1}{2} \int_\Sigma {}^{(\beta)}\pi^{aIJ} \delta^{(\beta)} A_{aIJ} + \frac{1}{\beta} \int_{\partial \Sigma} n^I \delta \tilde{s}_I + \delta\left(\ldots\right), \tag{16}$$

where $\approx$ means equality up to the simplicity constraint, leading to the conclusion that the usage of connection variables always leads to a boundary contribution to the symplectic structure. In fact, the boundary term results from the identity (The signs in (16) and (17) are compatible since s_a is chosen inward pointing in $\sum$.)

其中 $\approx$ 表示等式在忽略单纯性约束的意义下成立，由此可得结论: 使用联络变量总会给辛结构带来边界贡献。事实上，边界项来源于恒等式 ((16) 和 (17) 中的符号是相容的，因为在 $\sum$ 中 s_a 被选为内指向)

$$\frac{1}{2} {}^{(\beta)}\pi^{aIJ} \delta \Gamma^{\mathrm{H}}_{aIJ} - \frac{1}{\beta} \partial_a \left(E^{aI} \delta n_I\right) \approx \delta\left(\ldots\right), \tag{17}$$

which needs to be invoked in the passage to our connection variables. The total variation in (16) vanishes when computing the symplectic structure and thus the Poisson brackets.

该恒等式在过渡到我们的联络变量过程中必须用到。计算辛结构 (进而泊松括号) 时，(16) 中的总变分为零。

The spatial diffeomorphism constraint can be rewritten in terms of the connection variables as

空间微分同胚约束可以用联络变量改写为

$$V_a\left[N^a\right] = \int_\Sigma P^{ab}\mathcal{L}_N q_{ab} \approx \frac{1}{2}\int_\Sigma {}^{(\beta)}\pi^{aIJ}\mathcal{L}_N {}^{(\beta)}A_{aIJ} + \frac{1}{\beta}\int_{\partial\Sigma} n^I \mathcal{L}_N \tilde{s}_I. \tag{18}$$

The weak equality here is up to a term proportional to the "boost part" $n_I G^{IJ}$ of the Gauß constraint. It thus generates Lie derivatives on both the bulk and boundary variables. In order to compute the gauge transformations of the Gauß constraint, we have to partially integrate to free the canonical variables from derivatives, resulting in

此处的弱等式是忽略了与高斯约束“升压部分”$n_I G^{IJ}$ 成正比的项后的结果。因此该约束可以在体变量和边界变量上同时生成李导数。为了计算高斯约束的规范变换，我们需要通过分部积分将正则变量从导数中分离出来，最终得到

$$G^{IJ}\left[\lambda_{IJ}\right] = \int_\Sigma \lambda_{IJ} D_a {}^{(\beta)}\pi^{aIJ} = -\int_\Sigma {}^{(\beta)}\pi^{aIJ} D_a \lambda_{IJ} + \frac{2}{\beta}\int_{\partial\Sigma} \lambda_{IJ} n^I \tilde{s}^J. \tag{19}$$

Also here, we see that the local $\mathrm{SO}(D+1)$ gauge transformations act on both the bulk and boundary variables.

同样在这里，我们看到局域 $\mathrm{SO}(D+1)$ 规范变换会同时作用在体变量和边界变量上。

Holonomy-Flux Algebra

全纯-流代数

The quantum theory which we will derive from the above variables has strong similarities to lattice QCD, with the main difference that the lattice will become a dynamical object in the quantum theory, i.e., one works, in a certain precise sense, on all lattices simultaneously. We start by recalling that for a canonical quantization, we need to find a representation of a point-separating (unital) Poisson *-subalgebra of phase space functions on a Hilbert space. It is therefore our first task to find a subalgebra of phase space functions which admits such a representation. In making this choice, we are guided by the constraints of the theory, that is, we would like to have an easy transformation property of the algebra elements at least under $\mathrm{SO}(D+1)$ gauge transformations and spatial diffeomorphisms. Moreover, we would like to solve these constraints later on. The technical details of the following construction can be found in [46]; see also [21] for a short summary focusing on higher dimensions.

我们将从上述变量推导得到的量子理论与格点量子色动力学有诸多相似之处，主要区别在于格点在量子理论中会成为动力学对象，也就是说，从某种精确意义上说，我们同时在所有格点上开展研究。首先我们回顾，正则量子化要求我们在希尔伯特空间上找到相空间函数的点分离 (单位) 泊松 * 子代数的表示。因此我们的首要任务是找到能容纳该表示的相空间函数子代数。做此选择时，我们以理论的约束为指导，即我们希望至少在 $\mathrm{SO}(D+1)$ 规范变换和空间微分同胚下，代数元素具有简单的变换性质。此外，我们希望后续能求解这些约束。下述构造的技术细节可见文献 [46]；关于高维情况的简短综述可见 [21]。

From the $\mathrm{SO}(D+1)$ gauge transformations, it suggests itself to use holonomies of A_{aIJ} , since these can be used to form Wilson loops or, more generally, spin networks. We denote them by

从 $\mathrm{SO}(D+1)$ 规范变换出发，使用 A_{aIJ} 的全纯是自然的选择，因为我们可以用它构造威尔逊圈，更一般地还可以构造自旋网。我们将其记为

$$h_e^{\lambda}(A) = \mathcal{P}\exp\left(\int_e A_{aIJ}\tau_{\lambda}^{IJ}dx^a\right), \tag{20}$$

where e is a path (edge of a spin network) in $\sum, \mathcal{P}$ is the path-ordering symbol, and τ_{λ}^{IJ} are generators of $\mathrm{so}(D+1)$ in a representation labeled by $\lambda \in \mathbb{N}_0$ (We will see later on that the representations in the quantum theory are labeled by a single non-negative integer λ . We thus refrain from introducing proper notation such as $h_e^{\vec{\Lambda}}(A)$ for a holonomy in a general representation with the highest weight vector Λ .). All gauge-invariant information about the connection is encoded in the set of all holonomies. Next, we define the fluxes

其中 e 是一条路径 (自旋网的边)，位于 $\sum, \mathcal{P}$ 中，⍰ 是路径排序算符，τ_{λ}^{IJ} 是 $\mathrm{so}(D+1)$ 在由 $\lambda \in \mathbb{N}_0$ 标记的表示中的生成元 (我们后续会看到，量子理论中的表示由单个非负整数 λ 标记。因此我们没有为最高权向量为 Λ 的一般表示中的全纯引入 $h_e^{\vec{\Lambda}}(A)$ 这类规范记法。)。联络的所有规范不变信息都编码在全体全纯集合中。接下来，我们定义流:

$$\pi_S^n = \int_S n_{IJ}\pi^{aIJ}\varepsilon_{ab_1\ldots b_{D-1}}dx^{b_1}\wedge\ldots\wedge dx^{D-1} \tag{21}$$

generalizing the electric fluxes of Maxwell theory to a non-Abelian gauge group. n_{IJ} is a smearing function and S a $(D-1)$ -dimensional surface.

这是将麦克斯韦理论的电流量推广到非阿贝尔规范群。n_{IJ} 是抹函数，S 是 $(D-1)$ 维曲面。

Using (9), we can now compute the Poisson bracket between holonomies and fluxes. Exemplarily, in the case that e intersects S transversally in the point $p \in \sum$, we find

利用式 (9)，我们现在可以计算全纯和流之间的泊松括号。例如，当 e 与 S 在点 $p \in \sum$ 横截相交时，我们得到

$$\left\{h_e(A), \pi_S^n\right\} \propto \pm h_{e_1}(A)\tau^{IJ}n_{IJ}(p)h_{e_2}(A), \tag{22}$$

where $e = e_2 \circ e_1$ has been split at p . The sign in (22) depends on the intersection properties, that is, the orientation of e and S . Non-transversal and multiple intersections are detailed in [46], and we will neglect to discuss them here, as they are not required for the basic understanding of the quantization procedure. The Poisson bracket between two holonomies vanishes trivially, while the Poisson bracket between two fluxes cannot, in general, be expressed in terms of holonomies and fluxes again (This results from the singular smearing in terms of holonomies and fluxes; see [46].). This is however not an obstacle for the following quantization.

其中 $e = e_2 \circ e_1$ 已在 p 处拆分。式 (22) 中的符号取决于相交性质，即 e 和 S 的定向。非横截相交和多重相交的细节可见文献 [46]，由于理解基本量子化过程不需要这些内容，我们在此不做讨论。两个全纯之间的泊松括号平凡为零，而两个流之间的泊松括号一般无法再用全纯和流表示 (这源于全纯和流的奇异抹化；可见文献 [46])。但这对后续量子化并不构成障碍。

On $\partial \sum$, we consider the phase space functions

在 $\partial \sum$ 上，我们考虑相空间函数

$$L^{IJ} = \hat{s}_a \pi^{aIJ} = 2/\beta n^{[I}\tilde{s}^{J]}, \tag{23}$$

smeared over surfaces S in the same manner as (21). Using (16), their Poisson bracket reads

按式 (21) 的相同方式对曲面 S 做抹化处理。利用式 (16)，它们的泊松括号为

$$\left\{L_S^{IJ}, L_{S'}^{KL}\right\} = 4\delta^{J][K]}L_{S\cap S'}^{L][I}\text{ , or }\left\{L_S^{n}, L_{S'}^{n'}\right\} = -2L_{S\cap S'}^{[n,n']} \tag{24}$$

in terms of smearing functions n^{IJ} and n'^{IJ} . The boundary variables are thus just the restrictions of the fluxes to $\partial \sum$.

用抹函数 n^{IJ} 和 n'^{IJ} 表示。因此边界变量就是流在 $\partial \sum$ 上的限制。

We can now take the free algebra generated by all holonomies and fluxes and divide it by the commutator relations implied by (22) and its generalizations, that is, $[A, B] = \hbar/i\{A, B\}$. This gives us a unital *-algebra $\mathfrak{A}$, which we now have to represent on a Hilbert space.

我们现在可以取由全体全纯和流生成的自由代数，再除以由式 (22) 及其推广给出的对易关系，也就是 $[A, B] = \hbar/i\{A, B\}$ 。这就得到了单位 * 代数 $\mathfrak{A}$ ，我们接下来需要将它表示在希尔伯特空间上。

Outline of Quantization

量子化纲要

Most parts of the quantization procedure are analogous to the standard case in 3+1 dimensions up to replacing the gauge groups, and we will not detail them here. Rather, we focus on the new aspects introduced by the simplicity constraint and its interplay with the area operator.

除规范群需要替换外，量子化流程的大部分内容都与 3+1 维标准情形类似，因此本文在此不展开详述。我们将重点讨论由简约约束带来的新内容，以及它和面积算符的相互作用。

The Simplicity Constraint

简洁性约束

We now proceed to implement the simplicity constraints (8) as operator equations on the spin network basis states. Since only the smeared fluxes π_S^n , and not $\pi^{aIJ}(x)$, are well-defined operators, we have to regularize (8) in terms of fluxes. This has been done in detail in [21], and we recall the main results. The idea is to implement all of the constraints (8) at a given point $x \in \sum$ by smearing each π^{aIJ} over an arbitrary surface S containing x and then to compute the action of the operator

我们现在将着手把简洁性约束 (8) 实现为自旋网络基态上的算符方程。由于只有弥散通量 π_S^n 而非 $\pi^{aIJ}(x)$ 是良定义算符，我们必须用通量对 (8) 做正则化。这一步在文献 [21] 中已有详细推导，我们在此回顾主要结论。核心思路是: 在给定点 $x \in \sum$ 实现所有约束 (8)，我们先把每个 π^{aIJ} 弥散到包含 x 的任意曲面 S 上，再计算该算符的作用

$$\hat{\pi}_S^{[IJ]}\hat{\pi}_{S'}^{kl]} \tag{25}$$

on spin networks in the limit that the surfaces S and S' are shrunk to the point x . There are two nontrivial cases: x is an inner point of an edge e , or x is a vertex of the graph γ on which the spin network is defined. The implementation on a vertex discussed here has been put forward in [23], while the implementation on an edge follows directly from [47].

当曲面 S 和 S' 收缩至点 x 时，作用于自旋网络的极限结果。这里存在两种非平凡情况: x 是边 e 的内点，或是自旋网络所定义的图 γ 的顶点。本文讨论的顶点处约束实现方案由文献 [23] 提出，而边处的约束实现可直接由文献 [47] 得到。

In the first case, a generic (e is not contained in S or S' in a neighborhood of x ; otherwise, the action vanishes.) choice of surfaces yields the condition

第一种情况中，一般的曲面选择 (在 x 的邻域内 e 不包含于 S 或 S' ， 否则作用结果为零) 给出条件

$$\tau_{\vec{\Lambda}}^{[IJ]}\tau_{\vec{\Lambda}}^{KL]} = 0 \tag{26}$$

with $\tau_{\vec{A}}^{IJ}$ being the generators of so$(D+1)$ in an irreducible representation labeled by the highest weight vector $\vec{\Lambda}$. This result follows directly from (22) or visualizing the π^{aIJ} as derivative operators acting on the holonomies (20). It has been shown in [47] that the condition (26) on the generators of the Lie algebra so$(D+1)$ is equivalent to the restriction to a subset of representations with the highest weight vector $\vec{\Lambda} = (\lambda, 0, \dots, 0), \lambda \in \mathbb{N}_0$. In the mathematical literature, these representations are called spherical, most degenerate, (completely) symmetric, or representations of class 1 (with respect to an SO(D) subgroup). Within loop quantum gravity, they are usually denoted as ”simple” representation, which originates from their relation to simple bi-vectors through the simplicity constraint. This result now justifies the notation τ_λ^{IJ} used in this chapter, that is, to denote the representations by λ instead of $\vec{\Lambda}$.

其中 $\tau_{\vec{\Lambda}}^{IJ}$ 是最高权向量 $\vec{\Lambda}$ 标记的不可约表示中 $\mathrm{so}(D+1)$ 的生成元。该结果可直接由式 (22) 得到，也可通过将 π^{aIJ} 视为作用在整圈 (20) 上的导数算子直观理解。文献 [47] 已证明，李代数 $\mathrm{so}(D+1)$ 生成元满足的条件 (26) 等价于将表示限制为最高权向量满足 $\vec{\Lambda}=(\lambda,0,\ldots,0), \lambda\in\mathbb{N}_0$ 的子集。在数学文献中，这类表示被称为球面表示、最高退化表示、(完全) 对称表示，或相对于 $\mathrm{SO}(D)$ 子群的 1 类表示。在圈量子引力中，它们通常被称为“简单”表示，这一名称源于简洁性约束将其与简单双矢量关联起来。该结果正好为本章使用的记号 τ_{λ}^{IJ} 提供了合理性依据，即我们用 λ 而非 $\vec{\Lambda}$ 来标记这类表示。

The case of a vertex is more complicated, since the quantum simplicity constraints turn out to be non-commuting. Different strategies to deal with this problem in 3+1 dimensions have been put forward in the literature; see [23] for an overview. We will consider here the dimension-independent solution proposed in [23], which amounts to choosing a maximally commuting subset of vertex simplicity constraints, much in analogy with the choice of a maximal commuting subset of operators in quantum mechanics, or the strategy of gauge unfixing mentioned in section ”The Palatini Action.” Technically, one first finds a set of ”basic” simplicity constraints from considering all possible smearing surfaces, which turns out to be the most general set possible, that is, the two fluxes each acting only on a single edge incident at v [21]. Next, one proposes a commuting subset of vertex simplicity constraints and shows that adding a new vertex simplicity constraint spoils commutativity [23]. The result of this procedure is rather intuitive: in a given recouping scheme, which is in one-to-one correspondence with the choice of maximally commuting subset, the intertwining representations are all simple. We call such an intertwiner simple (with respect to a given recoupling scheme). We note that such an intertwiner is in general not simple with respect to another recoupling scheme, as follows from the general decomposition [48,49]

顶点的情况更复杂，因为量子简洁性约束是不对易的。文献中已经提出了多种处理 3+1 维下该问题的策略，综述可见 [23]。我们在这里讨论文献 [23] 提出的维度无关解，该方案的核心是选取顶点简洁性约束的极大对易子集，这非常类似于量子力学中选取极大对易算符子集的思路，也和“帕拉蒂尼作用量”一节提到的规范非固定策略一致。技术上，我们首先从所有可能的弥散曲面中得到一组“基本”简洁性约束，这其实是最一般的可能集合: 即两个通量都分别只作用在入射于 v 的单条边上 [21]。接下来，我们构造出顶点简洁性约束的一个对易子集，并证明再加入任意一个新的顶点简洁性约束都会破坏对易性 [23]。这个过程得到的结果相当直观: 在与极大对易子集一一对应的特定重耦合方案中，所有纠缠表示都是简单表示。我们称这样的 intertwiners(缠结算子) 为简单缠结算子 (相对于给定重耦合方案)。我们注意到，根据一般分解结果 [48,49]，这样的缠结算子相对于其他重耦合方案一般不是简单的

$$(\lambda_1,0,\ldots,0)\otimes(\lambda_2,0,\ldots,0)=\sum_{k=0}^{\lambda_2}\sum_{l=0}^{\lambda_2-k}(\lambda_1+\lambda_2-2k-l,l,0,\ldots,0)\ \ (\lambda_2\leq\lambda_1)$$

(27)

However, the dimension of the intertwiner space is independent of the choice of recoupling scheme, with respect to which the intertwiner is simple. We remark that there exists an intertwiner which is simple with respect to every recouping scheme, known as the Barrett-Crane intertwiner in $3+1$ dimensions [39]. Its $\mathrm{SO}(D+1)$ analog has been constructed in [47].

然而，缠结算子空间的维数与简单缠结算子所对应的重耦合方案选择无关。需要指出的是，存在一种对所有重耦合方案都满足简单性的缠结算子，即 $3+1$ 维中的巴雷特-克兰缠结算子 [39]，其 $SO(D+1)$ 类比物已在文献 [47] 中构造完成。

It has not been checked explicitly if the operation of the quantum Hamiltonian constraint preserves any chosen simple intertwiner or whether it needs to be modified to do so, e.g., by a gauge unfixing procedure.

目前尚未明确验证量子哈密顿约束的作用是否会保持任意选定的简单缠结算子，也未验证是否需要对其修改 (例如通过规范解除过程) 才能实现这一点。

The Area Operator

面积算符

We are now in the position to address the question of what happens to the geometrical properties of certain classical objects in the quantum theory. The so far best understood geometric operator in loop quantum gravity is the area operator, where area always refers to surface a spatial codimension 1. The area operator was originally proposed in [50]; see also [21] for an account tailored to our situation. We will shortly sketch its derivation and compute its action on holonomies.

我们现在可以探讨经典几何对象在量子理论中的几何性质变化问题。圈量子引力中目前研究最充分的几何算符就是面积算符，此处的面积始终指空间余维数为 1 的曲面的面积。面积算符最初在文献 [50] 中提出；适合本文研究场景的相关说明可见文献 [21]。下文将简要概述它的推导过程，并计算它对全纯移的作用。

Classically, the area of a surface S is given by

经典层面，曲面 S 的面积可表示为

$$A[S]=\int_S d^{D-1}x\sqrt{\det h_{\alpha\beta}}, \tag{28}$$

where $h_{\alpha\beta}, \alpha, \beta = 1, \ldots, D-1$ is the induced metric on S , as inherited from q_{ab} . We can rewrite this expression as

其中 $h_{\alpha\beta}, \alpha, \beta = 1, \ldots, D-1$ 是 S 上从 q_{ab} 诱导得到的度量。我们可以将该式改写为

$$A[S]=\int_S d^{D-1}x\sqrt{\hat{n}_a\hat{n}_b q q^{ab}}=\int_S d^{D-1}x\sqrt{\frac{\beta^2}{2}\hat{n}_a\hat{n}_b(\beta)\,\pi^{aIJ(\beta)}\pi^b{}_{IJ}}, \tag{29}$$

where $\hat{n}_a = \epsilon_{a\beta_1\ldots\beta_{D-1}}\epsilon^{\beta_1\cdots\beta_{D-1}}/(D-1)!$ is the properly densitized conormal on S . As a next step, we need to express the area in terms of fluxes. To do this, we note that the above integral is defined as the limit of Riemann sum approximations. Using the decomposition $S=\cup_i S_i$ and a set of points $x_i \in S_i$ and denoting by $|S_i|$ the coordinate volume of S_i , it follows that

其中 $\hat{n}_a = \varepsilon_{a\beta_1\ldots\beta_{D-1}}\varepsilon^{\beta_1\cdots\beta_{D-1}}/(D-1)!$ 是 S 上经过适当 densitization 的余法线。下一步，我们需要将面积用通量表示。为此我们注意到，上述积分由黎曼和逼近的极限定义，利用分解 $S = \cup_i S_i$ 、点集 $x_i \in S_i$ ，并用 $|S_i|$ 表示 S_i 的坐标体积，可得

$$A\left[S\right] = \lim_{S_i\to 0}\sum_i |S_i|\sqrt{\frac{\beta^2}{2}\hat{n}_a\left(x_i\right)\hat{n}_b(x_i)^{(\beta)}\pi^{aIJ}(x_i)^{(\beta)}\pi^b{}_{IJ}\left(x_i\right)}$$

$$= \lim_{S_i\to 0}\sum_i\sqrt{\frac{\beta^2}{2}\pi^{aIJ}_{S_i}\pi^b{}_{S_iIJ}}. \tag{30}$$

We can thus quantize the area operator by substituting flux operators in (30) and appealing to the spectral theorem to define the square root (The operator under the square root is essentially self-adjoint and non-negative [21,46].). We study the action of this operator on a holonomy of an edge e intersecting S transversally, as this is enough for, e.g., entropy computations. Splitting $e = e_2 \circ e_1$ at the intersection point, it follows that [21]

我们因此可以通过将 (30) 中的通量替换为通量算符，利用谱定理定义平方根 (平方根下的算符本质上是自伴随且非负的 [21,46])，从而对面积算符进行量子化。我们研究该算符在与 S 横截相交的边 e 的全纯移上的作用，这对于熵计算这类问题已经足够。将 $e = e_2 \circ e_1$ 在交点处拆分，可得 [21]

$$\widehat{A}\left[S\right]h^\lambda_e\left(A\right) = 8\pi G\hbar\sqrt{\beta^2}h^\lambda_{e_2}\left(A\right)\sqrt{-2\tau^{IJ}_\lambda\tau_{\lambda IJ}}h^\lambda_{e_1}\left(A\right)$$

$$= 8\pi G\hbar\sqrt{\beta^2}\sqrt{\lambda\left(\lambda + D - 1\right)}h^\lambda_e\left(A\right), \tag{31}$$

that is, the action of the operator is diagonal on holonomies and proportional to the quadratic Casimir of $\mathrm{SO}\left(D+1\right)$.

也就是说，该算符在全纯移上的作用是对角的，且与 $\mathrm{SO}\left(D+1\right)$ 的二次卡西米尔量成正比。

Brief Comparison in the Case $D = 3$

$D = 3$ 情形的简要对比

For $D = 3$, it is instructive to compare our results with the standard quantization based on Ashtekar-Barbero variables. The geometric structure of holonomies and fluxes is the same in both cases. The main difference are the gauge groups SU(2) and SO(4), respectively, as well as the additional simplicity constraint in the latter case. From the above discussion, we see that holonomies obeying the quantum simplicity constraint are labeled by a non-negative integer λ , which maps to $2j$ when compared to the SU(2) representation label j [21, 27] . In this case, the area operator eigenvalues agree up to a global constant that can be absorbed into β , the analog of the Barbero-Immirzi parameter in the dimension-independent case. With the same mapping applied to the recoupling spins, also the intertwiners in both cases can be mapped onto each other, showing that the kinematical structure is very similar [23].

对于 $D = 3$，将我们的结果与基于阿西特卡-巴贝罗变量的标准量子化结果进行对比颇具启发性。两种情形中，全纯和通量的几何结构完全一致，主要区别在于规范群分别为 SU(2) 和 SO(4)，且后者存在额外的简单性约束。从上述讨论可知，满足量子简单性约束的全由非负整数 λ 标记，该整数在与 SU(2) 表示标记 j [21, 27] 对比时映射为 $2j$。在此情形下，面积算符的本征值仅相差一个整体常数，该常数可被吸收到 β 中——β 是维数无关情形下巴贝罗-伊米里齐参数的类比量。对重耦合自旋应用相同映射后，两种情形下的缠结子也可互相映射，这表明它们的运动学结构非常相似 [23]。

On the other hand, a dynamical comparison has not been performed so far. For this, the action of qualitatively new terms in the Hamiltonian constraint that arise in order to retain a classical first-class algebra [19, 20] needs to be studied in detail.

另一方面，目前尚未完成动力学层面的对比。要开展这一对比，需要详细研究为保留经典第一类代数而出现的定性新项在哈密顿约束中的作用 [19, 20]。

Supersymmetry

超对称

In this section, we want to review the results of the papers [51-55] dealing with the question on how to combine LQG with the concept of supersymmetry (SUSY) playing a major role in supergravity (SUGRA) and superstring theory. Supersymmetry is a new kind of symmetry that arose in the context of the famous results of Coleman-Mandula [56] and Haag-Lopuszanski-Sohnius [57] who were looking for symmetries of interacting QFTs that can have a nontrivial mixture with spacetime symmetries. As a consequence, it follows that supersymmetry interrelates both bosonic and fermionic degrees of freedom, that is, both force and matter particles, and therefore seems to be a natural candidate for the search for a unified field theory. Trying to combine LQG with SUSY thus also brings LQG closer to ideas of unification. Another important motivation for following such a program is the desire to be able to build a bridge and find relations between the various approaches to quantum gravity. The first step to achieve this goal is to apply LQG techniques to supergravity. In [18, 24, 25] , the quantization of the (minimal) supersymmetric 11-dimensional SUGRA in the framework of LQG is performed using the connection variables in higher dimensions as discussed in the previous section. Here, we want to focus on (extended) SUGRA in four spacetime dimensions. To this end, we first review the Holst-MacDowell-Mansouri action of $\mathcal{N} = 1, 2$ AdS SUGRA in $D = 4$ and subsequently outline the quantization of the so-called SUSY constraint using standard LQG techniques. After that, an alternative approach will be discussed using chiral variables unraveling an enlarged gauge symmetry which allows to keep SUSY more manifest in the classical as well as in quantum theory. Based on this observation, a quantization of the theory will be proposed. Finally, applications in the context of quantum supersymmetric black holes will be discussed.

在本节中，我们将梳理文献 [51-55] 的研究成果，这些文献围绕如何将圈量子引力 (LQG) 与超对称 (SUSY) 结合这一问题展开研究，而超对称在超引力 (SUGRA) 和超弦理论中都发挥着核心作用。超对称是一种新型对称性，它诞生于 Coleman-Mandula[56] 与 Haag-Lopuszanski-Sohnius[57] 的著名研究成果背景下, 这些研究旨在寻找可与时空对称性发生非平凡结合的相互作用量子场论 (QFT) 的对称性。由此得到的结论是，超对称将玻色子自由度与费米子自由度——也就是力粒子与物质粒子——相互关联，因此它似乎是寻找统一场论的自然候选方向。尝试将 LQG 与超对称结合，也会让 LQG 更靠近统一理论的思路。开展该研究的另一个重要动机，是希望在不同量子引力研究方法之间搭建桥梁、建立关联。实现这一目标的第一步是将 LQG 方法应用于超引力。在 [18, 24, 25] 中，研究者利用上一节讨论的高维联络变量，在 LQG 框架下完成了 (最小) 超对称 11 维超引力的量子化。本文我们将聚焦四维时空的 (扩展) 超引力。为此，我们首先梳理 $\mathcal{N}=1,2$ 中 AdS 超引力在 $D=4$ 的 Holst-MacDowell-Mansouri 作用量，随后概述如何利用标准 LQG 方法对所谓超对称约束进行量子化。之后我们会讨论另一种采用手征变量的方法，该方法揭示了一种扩大的规范对称性，能让超对称在经典理论和量子理论中都更明晰。基于这一结论，我们会提出该理论的一种量子化方案。最后，我们会讨论该理论在量子超对称黑洞领域的应用。

The Holst-MacDowell-Mansouri Action for AdS Supergravity with Boundaries

带边界的 AdS 超引力霍尔斯特-麦克道尔-曼苏里作用量

The Holst-like extension of $\mathcal{N}$ -extended pure Poincaré SUGRA actions in $D=4$ has been discussed in [58]. Here, following [52,53], we want to derive it in the case of a nontrivial cosmological constant $\Lambda=-3/L^2$ with L the so-called anti-de Sitter radius using a geometric approach to SUGRA also known as the Castellani-D'Auria-Fré approach [59-63]. In mathematical terms, this is based on the concept of a so-called super Cartan geometry (see [52, 53, 55]). This also allows a discussion about the boundary theory and, in particular, its compatibility with local supersymmetry.

$\mathcal{N}$ 扩展纯庞加莱超引力作用量的类霍尔斯特推广已在文献 [58] 中于 $D=4$ 框架下讨论。本文遵循文献 [52,53]，将在非平凡宇宙学常数 $\Lambda=-3/L^2$ (对应反德西特半径 L) 的情形下，通过超引力的几何方法即卡斯特拉尼-多里亚-弗雷方法 [59-63] 推导该作用量。从数学角度而言，该方法基于超嘉当几何的概念 (参见 [52, 53, 55])。这也允许我们讨论边界理论，尤其是它与局域超对称性的兼容性问题。

Pure AdS (Holst-)supergravity with $\mathcal{N}$ -extended supersymmetry ($\mathcal{N}=1,2$) can be described in a geometric way in terms of a gauge field $\mathcal{A}$ also referred to as the super Cartan connection (In mathematical terms, this means that AdS (Holst-) supergravity for $\mathcal{N}=1,2$ is described in terms of a super Cartan geometry modeled on the super Klein geometry $\left(\mathrm{OSp}(\mathcal{N}\mid 4),\mathrm{Spin}^+(1,3)\times\mathrm{SO}(\mathcal{N})\right)$ (see [52, 53, 55] for more details).) which takes values in the supersymmetric generalization of the anti-de Sitter algebra given by the orthosymplectic superalgebra $\mathfrak{osp}(\mathcal{N}\mid 4)$. This superalgebra is generated by bosonic generators (P_I, M_{IJ}, T^{rs}) corresponding to infinitesimal spacetime translations, Lorentz transformations, and R -symmetry transformations, respectively, as well as $4\mathcal{N}$ fermionic generators (Q^r_α) which combine to $\mathcal{N}$ Majorana spinors. These satisfy the following graded commutation relations:

带有 $\mathcal{N}$ 扩展超对称的纯 AdS(霍尔斯特-) 超引力 ($\mathcal{N}=1,2$) 可以通过规范场 $\mathcal{A}$ 以几何方式描述，该规范场也被称为超嘉当联络 (数学上，这意味着 $\mathcal{N}=1,2$ 的 AdS(霍尔斯特-) 超引力由以超克莱因几何 $\left(\mathrm{OSp}(\mathcal{N}\mid 4), \mathrm{Spin}^{+}(1,3)\times \mathrm{SO}(\mathcal{N})\right)$ 为模型的超嘉当几何描述，更多细节见 [52, 53, 55])，其取值为直交辛超代数 $\mathfrak{osp}(\mathcal{N}\mid 4)$ 给出的反德西特代数的超对称推广。该超代数由分别对应无穷小时空平移、洛伦兹变换和 R 对称变换的玻色子生成元 (P_I, M_{IJ}, T^{rs}) ，以及 $4\mathcal{N}$ 费米子生成元 (Q^r_α) 生成，这些费米子生成元组合为 $\mathcal{N}$ 个马约拉纳旋量。它们满足如下阶化对易关系:

$$[M_{IJ}, Q^r_\alpha] = \frac{1}{2} Q^r_\beta (\gamma_{IJ})^\beta{}_\alpha \tag{32}$$

$$[P_I, Q^r_\alpha] = -\frac{1}{2L} Q^r_\beta (\gamma_I)^\beta{}_\alpha \tag{33}$$

$$[T^{pq}, Q^r_\alpha] = \frac{1}{2L}\left(\delta^{qr} Q^p_\alpha - \delta^{pr} Q^q_\alpha\right) \tag{34}$$

$$\left[Q^r_\alpha, Q^s_\beta\right] = \delta^{rs}\frac{1}{2}(C\gamma^I)_{\alpha\beta} P_I + \delta^{rs}\frac{1}{4L}(C\gamma^{IJ})_{\alpha\beta} M_{IJ} - C_{\alpha\beta} T^{rs} \tag{35}$$

The super Cartan connection can be decomposed in the following way:

超嘉当联络可分解为如下形式:

$$\mathcal{A} = e^I P_I + \frac{1}{2}\omega^{IJ} M_{IJ} + \frac{1}{2}\widehat{A}_{rs} T^{rs} + \Psi^\alpha_r Q^r_\alpha \tag{36}$$

with e^I, ω^{IJ} , and $\widehat{A}_{rs}$ the co-frame, spin connection, and $\mathfrak{so}(\mathcal{N})$ gauge field, respectively, as well as Ψ^α_r the spin-3/2 Rarita-Schwinger fields. This connection can be used in order to formulate a Yang-Mills-type action principle for Holst-supergravity. To this end, one introduces a β -deformed inner product $\langle \cdot \wedge \cdot \rangle_\beta$ on $\mathfrak{g} \equiv \mathrm{osp}(\mathcal{N}\mid 4)$ -valued 2-forms on the underlying spacetime manifold M with $\beta \in \mathbb{R}\setminus\{0\}$ the Barbero-Immirzi parameter via

其中 e^I, ω^{IJ} 、$\widehat{A}_{rs}$ 分别是余标架、自旋联络和 $\mathfrak{so}(\mathcal{N})$ 规范场，Ψ^α_r 是自旋 3/2 拉里塔-施温格场。该联络可用于为霍尔斯特超引力构造杨-米尔斯型作用量原理。为此，我们在底流形时空 M 上取值于 $\mathfrak{g} \equiv \mathrm{osp}(\mathcal{N}\mid 4)$ 的 2 形式上，通过巴贝罗-伊米里齐参数 $\beta \in \mathbb{R}\setminus\{0\}$ 引入了 β 变形内积 $\langle \cdot \wedge \cdot \rangle_\beta$ ，表达式为:

$$\langle \cdot \wedge \cdot \rangle_\beta : \Omega^2(M,\mathfrak{g}) \times \Omega^2(M,\mathfrak{g}) \to \Omega^4(M),\ (\omega,\eta) \mapsto \mathrm{str}\left(\omega \wedge \mathbf{P}_\beta \eta\right) \tag{37}$$

with "str" denoting the Ad-invariant supertrace on $\mathfrak{g}$ and $\mathbf{P}_\beta$ a β -dependent operator on $\Omega^2(M,\mathfrak{g})$ which, for instance, in the $\mathcal{N}=1$ case is given by

其中"str" 表示 $\mathfrak{g}$ 上的 Ad 不变超迹，$\mathbf{P}_\beta$ 是作用在 $\Omega^2(M,\mathfrak{g})$ 上依赖于 β 的算子，例如在 $\mathcal{N}=1$ 情形下其表达式为

$$\mathbf{P}_\beta := \underline{\mathbf{0}} \oplus \mathcal{P}_\beta \oplus \mathcal{P}_\beta \quad \text{with} \quad \mathcal{P}_\beta := \frac{\mathbb{1} + i\beta\gamma_5}{2\beta} \tag{38}$$

Using this inner product, the so-called Holst-MacDowell-Mansouri action of $\mathcal{N}$ -extended pure AdS Holst-supergravity takes the form

利用该内积，$\mathcal{N}$ 扩展纯 AdS 霍尔斯特超引力的霍尔斯特-麦克道尔-曼苏里作用量可写为如下形式

$$S_{\mathrm{H-MM}}^{\beta,\mathcal{N}}(\mathcal{A})=\frac{L^2}{\kappa}\int_M \langle F(\mathcal{A})\wedge F(\mathcal{A})\rangle_\beta \tag{39}$$

with $F(\mathcal{A}) = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}\wedge\mathcal{A}]$ the curvature of the super Cartan connection $\mathcal{A}$. Expanding (39) in the components of the curvature, it follows that this action indeed yields the Holst action of $\mathcal{N}$ -extended AdS supergravity in the bulk together with certain boundary terms. As shown in [53] based on arguments developed in [62,63] using standard variables, the boundary terms arising from (39) are indeed unique if one imposes local SUSY invariance of the full action at the boundary.

其中 $F(\mathcal{A}) = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}\wedge\mathcal{A}]$ 是超嘉当联络 $\mathcal{A}$ 的曲率。将 (39) 按曲率分量展开，可看出该作用量确实能得到体区中 $\mathcal{N}$ 扩展 AdS 超引力的霍尔斯特作用量，以及若干边界项。正如文献 [53] 基于文献 [62,63] 用标准变量推导的结论所示，若要求整个作用量在边界处满足局域 SUSY 不变性，那么由 (39) 得到的边界项确实是唯一的。

More precisely, in the Castellani-D'Auria-Fré approach, local SUSY transformations are regarded as infinitesimal superdiffeomorphisms ε along the odd directions of an underlying supermanifold satisfying $i_\varepsilon e^I = 0$. One can then show that the Lagrangian $\mathcal{L}_{\text{full}}$ of the action (39) is invariant under local SUSY transformations provided that, at the boundary ∂M , the condition [62]

更准确地说，在 Castellani-D'Auria-Fré 方法中，局域 SUSY 变换被视为底流形奇方向上的无穷小李超微分同胚 ε ，该超流形满足 $i_\varepsilon e^I = 0$ 。可以证明，只要在边界 ∂M 处满足文献 [62] 给出的条件，作用量 (39) 的拉格朗日量 $\mathcal{L}_{\text{full}}$ 就在局域 SUSY 变换下保持不变

$$(\iota_\varepsilon \mathcal{L}_{\text{full}})|_{\partial M} = 0 \tag{40}$$

is satisfied. It can then be shown that the boundary terms as arising from (39) are indeed the uniquely fixed by this condition.

该条件成立后，即可证明由 (39) 得到的边界项确实由该条件唯一确定。

Quantization of the SUSY Constraint

超对称约束的量子化

The canonical analysis of the Holst action of $\mathcal{N} = 1$ SUGRA in $D = 4$ in the case of a vanishing cosmological constant has been first discussed in [64,65]. In the case of higher dimensions using the Ashtekar-Barbero-type variables as introduced in section "Higher Dimensions," this has been considered in [18, 24, 25] . Here, we follow [54] where a consistent treatment of the reality conditions of the fermionic variables has been included. Expanding (39) in the case $\mathcal{N} = 1$, it follows that the action modulo boundary terms takes the form

宇宙学常数为零情形下，$\mathcal{N}=1$ 维 $D=4$ 超引力 Holst 作用量的正则分析最早在文献 [64,65] 中讨论。若推广到更高维度，采用“高维”小节引入的 Ashtekar-Barbero 型变量，该问题已在 [18, 24, 25] 中被研究。本文遵循文献 [54] 的处理，其中包含了对费米子变量实性条件的自洽处理。将 (39) 在 $\mathcal{N}=1$ 情形下展开，可得忽略边界项后的作用量形式为

$$S_{\mathrm{H-MM}}^{\beta,\mathcal{N}=1}=\frac{1}{2\kappa}\int_{M}\mathrm{d}^4x\left(ee_I^{\mu}e_J^{\nu}\left[F(\omega)_{\mu\nu}^{IJ}-\frac{1}{2\beta}\varepsilon^{IJ}{}_{KL}F(\omega)_{\mu\nu}^{KL}\right]\right.$$

$$\left.+2\varepsilon^{\mu\upsilon\rho\sigma}\overline{\psi}_{\mu}\gamma_{\sigma}\mathcal{P}_{\beta}D_{\upsilon}^{(\omega)}\psi_{\rho}-e\frac{1}{L}\overline{\psi}_{\mu}\gamma^{\mu\upsilon}\psi_{\upsilon}+\frac{6}{L^2}e\right) \tag{41}$$

where $e := \det\left(e_{\mu}^{I}\right)$. By performing a 3+1 decomposition, i.e., by assuming that the spacetime M is globally hyperbolic and thus accordingly splits in the form $M = \mathbb{R}\times\sum$ with $\sum$ a $3D$ spacelike Cauchy hypersurface, one finds that (41) can be written in the following way:

其中 $e := \det\left(e_{\mu}^{I}\right)$。通过进行 3+1 分解，即假设时空 M 是整体双曲的，因此可分解为 $M = \mathbb{R}\times\sum$ 的形式，其中 $\sum$ 是 $3D$ 类空柯西面，可发现 (41) 可写为如下形式:

$$S_{\mathrm{H-MM}}^{\beta,\mathcal{N}=1}=\int_{\mathbb{R}}\mathrm{d}t\int_{\sum}\mathrm{d}^3x\left(\frac{1}{\kappa\beta}E_i^a\mathcal{L}_T{}^{\beta}A_a^i-\pi^a\mathcal{L}_T\psi_a+A_t^iG_i+N^aH_a+\overline{\psi}_tS+NH\right)$$

(42)with G_i, H_a , and H the Gauß, vector, and Hamilton constraint, respectively, as well as an additional constraint S called the SUSY constraint generating local SUSY transformation on the phase space. The phase space is generated by the gravitational electric field $E_a^i = \sqrt{q}e_a^i$ with q the induced metric on $\sum$ and the Ashtekar-Barbero connection ${}^{\beta}A_a^i = \Gamma_a^i+\beta K_a^i$ as well as the pair (ψ_a, π^a) consisting of the pullback of the Rarita-Schwinger field ψ_a to $\sum$ together with its canonically conjugate momentum π^a fixed by the reality condition

其中 G_i, H_a ，H 分别是高斯约束、矢量约束和哈密顿约束，此外还有一个额外约束 S ，称为超对称约束，它在相空间上生成局域超对称变换。相空间由引力电场 $E_a^i=\sqrt{q}e_a^i$ (其中 q 是 $\sum$ 上的诱导度规)、Ashtekar-Barbero 联络 ${}^{\beta}A_a^i = \Gamma_a^i+\beta K_a^i$ ，以及配对 (ψ_a, π^a) 构成，(ψ_a, π^a) 由拉里塔-施温格场 ψ_a 拉回至 $\sum$ 得到，其正则共轭动量 π^a 由实性条件固定

$$\Omega^a := \pi^a+\kappa^{-1}\varepsilon^{abc}\overline{\psi}_b\gamma_c\mathcal{P}_{\beta}=0 \tag{43}$$

The reality condition (43) is quite complicated due to its explicit dependence on the spatial metric. Thus, to simplify it, one introduces new canonically conjugate fermionic variables by setting [64]

由于实性条件 (43) 显式依赖空间度规，形式十分复杂。因此为简化该条件，研究者通过如下定义引入了新的正则共轭费米子变量 [64]

$$\phi_i=\frac{\sqrt[4]{q}}{\sqrt{\kappa}}e_i^a\psi_a \quad \text{and} \quad \pi_{\phi}^i=\frac{\sqrt{\kappa}}{\sqrt[4]{q}}e_a^i\pi^a \tag{44}$$

As can be checked directly, this indeed provides a canonical transformation on the phase space by re-defining the Ashtekar-Barbero connection according to

可以直接验证，通过按如下方式重新定义 Ashtekar-Barbero 联络，上述操作确实给出了相空间上的一个正则变换

$$ {}^{\beta}A^i_a \to {}^{\beta}A'^i_a = \Gamma^i_a + \beta K'^i_a \quad \text{with} \quad K'^i_a = K^i_a + \frac{\kappa}{\sqrt{q}}\epsilon^{ijk}e^l_a\overline{\phi}_j\gamma_k\mathcal{P}_\beta\phi_l \tag{45} $$

As a consequence, this yields the new canonically conjugate pairs $\left(A'^i_a, E^a_i\right)$ and $\left(\phi_i, \pi^i_\phi\right)$ with the non-vanishing Poisson brackets

由此可得新的正则共轭对 $\left(A'^i_a, E^a_i\right)$ 和 $\left(\phi_i, \pi^i_\phi\right)$，其非零泊松括号为

$$ \left\{{}^{\beta}A'^i_a(x), E^b_j(y)\right\} = \kappa\beta\delta^{(3)}(x,y) \quad \text{and} \quad \left\{\phi^\alpha_i(x), \pi^j_{\phi\beta}(y)\right\} = -\delta^j_i\delta^\alpha_\beta\delta^{(3)}(x,y) $$

(46)

In the new variables, the reality condition (43) takes the simplified form

在新变量下，实性条件 (43) 变为简化形式

$$ \Omega^i := \pi^i_\phi + \epsilon^{ijk}\overline{\phi}_j\gamma_k\mathcal{P}_\beta = 0 \tag{47} $$

and thus no longer depends on the spatial metric. As shown in [24,54], this allows to canonically quantize the theory adapting standard tools of LQG [7]. The constraints of the theory also have to be re-expressed in terms of the new variables. For instance, for the SUSY constraint, one finds

因此它不再依赖空间度规。正如文献 [24,54] 所示，这使得我们可以采用圈量子引力 [7] 的标准工具对该理论进行正则量子化。理论的约束也需要重新用新变量表示。例如，对于超对称约束，可以得到

$$ \begin{aligned} S = {} & i\epsilon^{abc}e^i_a\gamma_i\gamma_* D^{(\beta' A')}_b\left(\frac{1}{\sqrt[4]{q}}e^j_c\phi_j\right) + \frac{1}{\sqrt[4]{q}}\epsilon^{abc}e^l_c P_\beta\gamma_k\left(D^{(\beta')}_a e^k_b\right)\phi_l \\ & + \frac{\kappa}{\sqrt[4]{q}}\epsilon^{ijk}\gamma^l\phi_{[l}\left(\overline{\phi}_{i]}\mathcal{P}_\beta\gamma_k\phi_j\right) + \frac{\kappa\beta}{2\sqrt[4]{q}}\gamma_0\phi^i\left(\overline{\phi}_j\gamma_k\mathcal{P}_\beta\gamma_i\gamma^{(j}\phi^{k)}\right) \\ & + \frac{\kappa}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\epsilon^{jkl}\overline{\phi}_j\gamma_0\mathcal{P}_\beta\gamma_k\gamma_i\phi_l\right) - \frac{\sqrt[4]{q}}{L}\gamma^{0i}\phi_i \end{aligned} \tag{48} $$

In the canonical formulation of SUGRA, the SUSY constraint S plays a major role. This is due to the fact the Poisson bracket $\{S, S\}$ turns out to be proportional to the Hamiltonian constraint [64] (see also [66] in the context of the chiral theory as well as [67] in the context of symmetry reduced models). Hence, the SUSY constraint is superior to the Hamiltonian constraint in the sense that finding solution of the Hamiltonian constraint operator and, thus, solving the dynamics in the quantum theory amount to finding solutions of the corresponding SUSY constraint operator. This underlines the important role of the SUSY constraint in canonical SUGRA and the necessity to study its implementation in the quantum theory. Besides this, however, the SUSY constraint operator might also be of pure academic interest due to its strong connection to the

Hamiltonian constraint via the constraint algebra which may also fix some of the quantization ambiguities. In fact, this has been shown explicitly in the context of symmetry reduced models [67].

在超引力的正则表述中，超对称约束 S 发挥着核心作用。这是因为泊松括号 $\{S,S\}$ 正比于哈密顿约束 [64](另见手征理论背景下的 [66] 以及对称性约化模型背景下的 [67])。因此，超对称约束比哈密顿约束更根本: 在量子理论中，寻找哈密顿约束算符的解、进而求解动力学，等价于寻找对应超对称约束算符的解。这凸显了超对称约束在正则超引力中的重要地位，也说明必须研究它在量子理论中的实现方式。此外，由于超对称约束通过约束代数与哈密顿约束紧密关联，这种关联甚至可以固定部分量子化歧义，因此超对称约束算符也具有纯粹的理论研究价值，这一点已经在对称性约化模型的背景下被明确证明 [67]。

For the rest of this section, let us briefly outline the quantization of the theory as well as the implementation of the SUSY constraint operator. For more details, we refer the interested reader to [54]. As shown in [24], in order to solve the reality condition (47), it is worthwhile to enlarge the phase space by decomposing the fermionic variables in the form $\phi_i = \rho_i + \frac{1}{3}\gamma_i\sigma$ with $\sigma := \gamma^i\phi_i$ and ρ_i the trace-free part of ϕ_i defined via the secondary constraint $\Xi := \gamma^i\rho_i = 0$. Using this constraint together with the reality condition (47), one finds that (ρ_i, σ) satisfy the following non-vanishing Dirac brackets [54]:

在本节剩余部分，我们将简要概述该理论的量子化以及超对称约束算符的实现。感兴趣的读者可参阅文献 [54] 了解更多细节。正如文献 [24] 所示，为求解实条件 (47)，值得通过将费米子变量分解为 $\phi_i = \rho_i + \frac{1}{3}\gamma_i\sigma$ 的形式来扩充相空间，其中 $\sigma := \gamma^i\phi_i$ 和 ρ_i 是通过次级约束 $\Xi := \gamma^i\rho_i = 0$ 定义的 ϕ_i 的无迹部分。利用该约束与实条件 (47)，可得 (ρ_i, σ) 满足以下非零狄拉克括号 [54]:

$$\left\{\rho_i^\alpha(x), \rho_j^\beta(y)\right\}_{\mathrm{DB}} = i\mathbf{P}_{ij}^{\alpha\beta}\delta^{(3)}(x,y) \text{ and } \left\{\sigma^\alpha(x), \sigma^\beta(y)\right\}_{\mathrm{DB}} = -\frac{9i}{2}\delta_{ij}\delta^{\alpha\beta}\delta^{(3)}(x,y)$$

(49)

with

其中

$$\mathbf{P}_{ij}^{\alpha\beta} := \delta_{ij}\delta^{\alpha\beta} - \frac{1}{3}\left(\gamma_i\gamma_j\right)^{\alpha\beta} \tag{50}$$

the projection onto the subspace of trace-free Rarita-Schwinger fields. Due to the fact that $\mathbf{P}_{ij}^{\alpha\beta}$ defines a projection operator allows to quantize the fermionic fields by adapting standard tools of LQG as developed in [7]. Hence, by applying a specific regularization procedure, to ρ_i and σ , one associates operators $\hat{\theta}_i^{(\rho)}(x) \equiv \mathbf{P}_{ij}\hat{\theta}_j^{(\rho)}(x)$ and $\hat{\theta}^{(\sigma)}(x)$ localized at each point $x \in \sum$ acting on a Hilbert space $\mathfrak{H}_x$ of certain Grassmann-valued functions on a supermanifold. These operators satisfy the following the anticommutation relations:

是到无迹拉里塔-施温格尔场子空间的投影。由于 $\mathbf{P}_{ij}^{\alpha\beta}$ 定义了一个投影算符，我们可以通过适配文献 [7] 中发展的圈量子引力标准工具来量子化费米场。因此，通过对 ρ_i 和 σ 应用特定正则化方案，我们得到了定域在每个点 $x \in \sum$ 上的算符 $\hat{\theta}_i^{(\rho)}(x) \equiv \mathbf{P}_{ij}\hat{\theta}_j^{(\rho)}(x)$ 和 $\hat{\theta}^{(\sigma)}(x)$，这些算符作用在超流形上某种格拉斯曼值函数的希尔伯特空间 $\mathfrak{H}_x$ 上。这些算符满足下列反对易关系:

$$\left[\hat{\theta}_i^{(\rho)}(x), \hat{\theta}_j^{(\rho)}(y)\right] = \hbar\mathbf{P}_{ij}\delta_{x,y} \text{ and } \left[\hat{\theta}^{(\sigma)}(x), \hat{\theta}^{(\sigma)}(y)\right] = \frac{9\hbar}{2}\mathbb{1}\delta_{x,y} \tag{51}$$

The quantized Rarita-Schwinger field on $\mathfrak{H}_x$ is then defined via

$\mathfrak{H}_x$ 上量子化的拉里塔-施温格尔场由下式定义

$$\hat{\theta}_i(x) := \hat{\theta}_i^{(\rho)}(x) + \frac{1}{3}\gamma_i \hat{\theta}^{(\sigma)}(x) \tag{52}$$

The full Hilbert space $\mathfrak{H}_f$ of the quantized fermionic degrees of freedom is defined as the inductive limit of the tensor product Hilbert spaces $\mathfrak{H}_{\{x_1,\ldots,x_k\}} := \otimes_{i=1}^k \mathfrak{H}_{x_i}$ associated with a finite collection of points $\{x_1, \ldots, x_k\}$ in $\sum$.

量子化费米子自由度的全希尔伯特空间 $\mathfrak{H}_f$ 定义为与 $\sum$ 中有限点集 $\{x_1, \ldots, x_k\}$ 关联的张量积希尔伯特空间 $\mathfrak{H}_{\{x_1,\ldots,x_k\}} := \otimes_{i=1}^k \mathfrak{H}_{x_i}$ 的归纳极限。

The bosonic degrees of freedom given by the transformed Ashtekar-Barbero connection (45) and its canonically conjugate momentum E_i^a are quantized in the standard way by studying the associated holonomies and electric fluxes leading to a Hilbert space $\mathfrak{H}_{\text{grav}}$ generated by SU(2) spin network states. As the total Hilbert space $\mathfrak{H}^{\text{LQSG}}$ of loop quantum supergravity (LQSG), we thus have

由变换后的阿西特卡-巴贝罗联络 (45) 及其正则共轭动量 E_i^a 给出的玻色子自由度可按标准方法量子化: 研究相关的和乐与电通量，得到由 SU(2) 自旋网络态张成的希尔伯特空间 $\mathfrak{H}_{\text{grav}}$ 。因此，圈量子超引力 (LQSG) 的总希尔伯特空间 $\mathfrak{H}^{\text{LQSG}}$ 为

$$\mathfrak{H}^{\text{LQSG}} = \mathfrak{H}_{\text{grav}} \otimes \mathfrak{H}_f \tag{53}$$

On this Hilbert space, we finally want to implement the SUSY constraint

最后，我们要在该希尔伯特空间上实现超对称约束

$$S[\eta] := \int_{\sum} \mathrm{d}^3 x \bar{\eta} S = \sum_{i=1}^{6} S^{(i)}[\eta] \tag{54}$$

which, according to (48), splits into six subcomponents $S^{(i)}[\eta], i = 1, \ldots, 6$. As argued in [54], the form (48) of the SUSY constraint is preferred due to its simplicity and the fact that it does not explicitly involve the extrinsic curvature which, by Thiemann's prescription [68], is related to the Hamiltonian constraint via the Poisson bracket which should be avoided. However, this then requires an alternative quantization procedure. It has been shown in [54] that all the components $S^{(1)}[\eta]$ can indeed be implemented rigorously and consistently on $\mathfrak{H}^{\text{LQSG}}$. In what follows, let us only comment briefly on the first one $S^{(1)}[\eta]$. By choosing a triangulation of $\sum$ similar as in [68], one finds that the quantum analog of $S^{(1)}[\eta]$ schematically takes the form (for more details, we refer to [54])

根据式 (48)，该约束可分解为六个子分量 $S^{(i)}[\eta], i = 1, \ldots, 6$。正如文献 [54] 所述，超对称约束采用形式 (48) 更优，因为它形式简洁，且不显含外曲率——根据 Thiemann 的方案 [68]，外曲率通过泊松括号与哈密顿约束关联，应当避免出现在约束形式中。但这就需要采用替代量子化方案。文献 [54] 已证明，所有分量 $S^{(1)}[\eta]$ 确实可以在 $\mathfrak{H}^{\mathrm{LQSG}}$ 上被严格且自洽地实现。下文仅对第一个分量 $S^{(1)}[\eta]$ 作简要说明。对 $\sum$ 采用与文献 [68] 类似的三角剖分后可以发现，$S^{(1)}[\eta]$ 的量子对应形式上可写为 (更多细节参见文献 [54])

$$\hat{S}^{(1)}[\eta] := -\frac{1}{3\hbar^2\kappa^2}\sum_{v\in V(\gamma)}\frac{8}{E(v)}\overline{\eta}(x_i)\, i\varepsilon^{IJK}\gamma_j\gamma_*\left[\widehat{\mathcal{X}}_K(s_J(\Delta)) - \widehat{\mathcal{X}}_K(x)\right]\mathrm{tr}\left(\tau_j h_{s_K(\Delta)}\left[h^{-1}_{s_I(\Delta)}, \hat{V}_v\right]\right) \tag{55}$$

with

其中

$$\widehat{\mathcal{X}_K}(s_J(\Delta)) := \mathrm{tr}\left(\tau_k h_{s'_K(\Delta)}\left[h^{-1}_{s'_K(\Delta)}, \sqrt{\hat{V}}_{s_J(\Delta)}\right]\right) H_{s_J(\Delta)}\hat{\theta}_k(s_J(\Delta)) \tag{56}$$

As explained in [54], in order for the difference $\widehat{\mathcal{X}_K}(s_J(\Delta)) - \widehat{\mathcal{X}_K}(x)$ to be nontrivial and thus to correspond to a true quantum analog of the regularized covariant derivative, for $\hat{V}_v$, one needs to take the Rovelli-Smolin variant of the volume operator $\hat{V}_v \equiv \hat{V}_v^{\mathrm{RS}}$ [69-71] as this operator does not act trivially on coplanar vertices in general. However, by choosing different forms of the SUSY constraint, also other quantizations have been proposed in [54] using the Ashtekar-Lewandowski volume operator.

正如文献 [54] 所述，为了让差值 $\widehat{\mathcal{X}_K}(s_J(\Delta)) - \widehat{\mathcal{X}_K}(x)$ 非平凡，从而对应正则化协变导数的真实量子对应，对 $\hat{V}_v$ 而言需要采用体积算符 $\hat{V}_v \equiv \hat{V}_v^{\mathrm{RS}}$ 的 Rovelli-Smolin 变种 [69-71]，因为该算符一般来说对共面顶点作用非平凡。不过，文献 [54] 也通过选取不同形式的超对称约束，提出了其他量子化方案，这类方案使用 Ashtekar-Lewandowski 体积算符。

It follows from (55) and (56) that the SUSY constraint operator, while acting on a spin network state, creates new vertices coupled to fermions. Hence, physical solutions of this operator necessarily need to contain both bosonic and fermionic degrees of freedom which is consistent with SUSY. The specific form of such solutions, however, has to be investigated in much more detail in the future. Also, it has to be checked whether the constraint algebra is anomaly-free and how this relates to the standard quantization scheme of the Hamiltonian constraint.

由式 (55) 和 (56) 可知，超对称约束算符作用在自旋网络态上时，会产生与费米子耦合的新顶点。因此，该算符的物理解必然同时包含玻色自由度和费米自由度，这与超对称的要求一致。不过这类解的具体形式仍有待未来更深入的研究。此外还需要验证约束代数是否无反常，以及它和哈密顿约束的标准量子化方案存在何种关联。

The Chiral Theory and Manifest SUSY

手征理论与显式超对称

As seen in the previous section, the SUSY constraint operator in LQG takes a very complicated form. In fact, it acquires an even more complex structure than the ordinary Hamiltonian constraint operator in the bosonic theory. This makes the finding of its solutions very difficult and thus the study of the dynamics in the quantum theory. Moreover, this makes this theory almost inaccessible for direct physical applications such as in the context of black holes or cosmology.

正如上一节所见，圈量子引力 (LQG) 中的超对称 (SUSY) 约束算符形式极为复杂，实际上它比玻色子理论中的普通哈密顿约束算符结构还要复杂。这导致我们极难找到它的解，进而阻碍了量子理论动力学的研究。此外，这种复杂性使得该理论几乎无法直接用于物理应用，比如黑洞或宇宙学场景。

Interestingly, this situation changes drastically in the case of the chiral theory. In fact, the SUSY constraint turns out to be much simpler in this case. In particular, in [72], it has been observed that there exists some remnant supersymmetry in the constraint algebra. Based on this observation, in [73-75], a quantization has been proposed that allows to keep supersymmetry more manifest in the theory. So far, the construction of the quantum theory remained rather formal. Moreover, the (geometric) origin of the hidden supersymmetry in the constraint algebra as well as possible generalizations to include real values of the Barbero-Immirzi parameter, the extended supersymmetry, as well as the boundary theory remained unclear. In the following, we review the results of [51-53, 76] where all these questions are addressed in a unified geometric way, starting from the Holst-MacDowell-Mansouri action of AdS SUGRA (39). We then outline the construction of the quantum theory including SUSY spin nets and area operator and finally discuss applications to black holes and cosmology.

有趣的是，在手征理论中情况发生了极大转变: 超对称约束在这里实际上简化了很多。特别地，文献 [72] 已观测到约束代数中存在残余超对称。基于这一观测，文献 [73-75] 提出了一种量子化方案，能让超对称在理论中保持得更为显式。到目前为止，该量子理论的构造仍然相当形式化。此外，约束代数中隐藏超对称的 (几何) 起源、容纳巴贝罗-伊米里兹参数实数值的可能推广、扩展超对称以及边界理论，这些问题都仍不明确。下文我们将综述文献 [51-53, 76] 的研究成果，这些成果从反德西特超引力 (AdS SUGRA) 的霍尔斯特-麦克道威尔-曼苏里作用量 (39) 出发，用统一的几何方法处理了上述所有问题。随后我们会概述包含超对称自旋网和面积算符的量子理论构造，最后讨论该理论在黑洞和宇宙学中的应用。

The Chiral Holst-MacDowell-Mansouri Action

手性霍尔斯特-麦克道尔-曼苏里作用量

In general, the operator $\mathbf{P}_\beta$ contained in the definition (37) of the β -deformed inner product does not commute with local $\mathrm{OSp}(\mathcal{N} \mid 4)$ gauge transformations as, for instance, it acts trivially on gauge transformations corresponding to infinitesimal spacetime translations. Thus, the Holst-MacDowell-Mansouri action, in general, is not manifestly invariant under $\mathrm{OSp}(\mathcal{N} \mid 4)$ gauge transformations. Since, by (35), the anticommutator of two fermionic generators of opposite chirality generates infinitesimal spacetime translations, it follows that the action can be manifestly invariant under some remnant supersymmetry iff $\mathbf{P}_\beta$ singles out fermionic fields of specific chirality. Obviously, this happens only when the Barbero-Immirzi parameter takes the values $\beta = \pm i$. In fact, in this case, it follows that the operator $\mathbf{P}_{-i}$ decomposes in the form $\mathbf{P}_{-i} = \widetilde{\mathbf{P}}_{-i} \circ \mathbf{P}^{\mathfrak{osp}(\mathcal{N}|2)}$ with $\mathbf{P}^{\mathfrak{osp}(\mathcal{N}|2)} : \mathfrak{osp}(\mathcal{N} \mid 4) \to \mathfrak{osp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ the projection operator onto the (complexified) chiral subsuperalgebra $\mathfrak{osp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ of $\mathfrak{osp}(\mathcal{N} \mid 4)$. Applying this projection operator onto the super Cartan connection (36), this yields a $\mathfrak{osp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ gauge field

一般而言, 包含在 β 变形内积定义式 (37) 中的算符 $\mathbf{P}_\beta$ 与局域 $\mathrm{OSp}(\mathcal{N} \mid 4)$ 规范变换不对易, 例如, 它对对应无穷小时空平移的规范变换作用平凡。因此，一般情况下，霍尔斯特-麦克道尔-曼苏里作用量并不具备明显的 $\mathrm{OSp}(\mathcal{N} \mid 4)$ 规范变换不变性。根据式 (35)，两个手性相反的费米子生成元的反对易子生成无穷小时空平移，由此可得: 作用量能具有明显的剩余超对称性，当且仅当 $\mathbf{P}_\beta$ 筛选出特定手性的费米场。显然, 仅当巴贝罗-伊米里齐参数取值为 $\beta = \pm i$ 时才能满足这一条件。实际上, 在此情形下可推得算符 $\mathbf{P}_{-i}$ 可分解为 $\mathbf{P}_{-i} = \widetilde{\mathbf{P}}_{-i} \circ \mathbf{P}^{\mathfrak{osp}(\mathcal{N}|2)}$ 的形式, 其中 $\mathbf{P}^{\mathfrak{osp}(\mathcal{N}|2)} : \mathfrak{osp}(\mathcal{N} \mid 4) \to \mathfrak{osp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ 是到 $\mathfrak{osp}(\mathcal{N} \mid 4)$ 的 (复化) 手性超子代数 $\mathfrak{osp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ 的投影算符。将该投影算符作用于超嘉当联络 (36)，我们就得到一个 $\mathfrak{osp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ 规范场

$$\mathcal{A}^+ := \mathbf{P}^{\mathrm{osp}(\mathcal{N}|2)}\mathcal{A} = A^{+i}T_i^+ + \psi_r^A Q_A^r + \frac{1}{2}\widehat{A}_{rs}T^{rs} \tag{57}$$

containing the bosonic self-dual Ashtekar connection $A^{+i} = \Gamma^i - iK^i$ for which reason it is also referred to as the super Ashtekar connection. Using super Ashtekar connection, it follows that the Holst-MacDowell-Mansouri action in the chiral limit takes the intriguing form

它包含玻色型自对偶阿西特卡联络 $A^{+i} = \Gamma^i - iK^i$ ，因此也被称为超阿西特卡联络。利用超阿西特卡联络，可以推得手性极限下霍尔斯特-麦克道尔-曼苏里作用量呈现出十分有趣的形式

$$S^{\beta=-i}_{\mathrm{H-MM}}(\mathcal{A}) = \frac{i}{\kappa}\int_M \langle F(\mathcal{A}^+) \wedge \mathcal{E}\rangle + \frac{1}{4L^2}\langle \mathcal{E} \wedge \mathcal{E}\rangle + S_{\mathrm{bdy}}(\mathcal{A}^+) \tag{58}$$

with $\mathcal{E}$ the super electric field canonically conjugate to $\mathcal{A}^+$ and transforming under the adjoint representation of $\mathrm{OSp}(\mathcal{N} \mid 2)_{\mathbb{C}}$. The boundary action $S_{\mathrm{bdy}}(\mathcal{A}^+)$ of the theory is given by

其中 $\mathcal{E}$ 是与 $\mathcal{A}^+$ 正则共轭的超电场，并按 $\mathrm{OSp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ 的伴随表示变换。该理论的边界作用量 $S_{\mathrm{bdy}}(\mathcal{A}^+)$ 由下式给出

$$S_{\mathrm{bdy}}(\mathcal{A}^+) \equiv S_{\mathrm{CS}}(\mathcal{A}^+) = \frac{k}{4\pi}\int_H \left\langle \mathcal{A}^+ \wedge \mathrm{d}\mathcal{A}^+ + \frac{1}{3}\mathcal{A}^+ \wedge [\mathcal{A}^+ \wedge \mathcal{A}^+]\right\rangle \tag{59}$$

with $H := \partial M$ and thus, in particular, corresponds to the action of a $\mathrm{OSp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ super Chern-Simons theory with (complex) Chern-Simons level $k = i4\pi L^2/\kappa = -i12\pi/\kappa\Lambda$. According to the discussion at the end of section "The Holst-MacDowell-Mansouri Action for AdS Supergravity with Boundaries," it follows that this boundary action as arising from (39) in the chiral limit is unique if one imposes supersymmetry invariance at the boundary (see also [52, 53]). Moreover, condition (40) implies that the chiral projection $\mathbf{P}_{-i}F(\mathcal{A})$ of the curvature has to vanish at the boundary which turns out to be equivalent to the boundary condition

对于 $H := \partial M$ ，因此尤其对应于 (复) 陈-西蒙斯水平为 $k = i4\pi L^2/\kappa = -i12\pi/\kappa\Lambda$ 的 $\mathrm{OSp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ 超陈-西蒙斯理论的作用量。根据末尾章节“带边界的 AdS 超引力的 Holst-MacDowell-Mansouri 作用量”中的讨论可知: 若要求边界满足超对称性不变性，由手征极限下式 (39) 得到的该边界作用量是唯一的 (另见 [52, 53])。此外，条件 (40) 表明曲率的手征投影 $\mathbf{P}_{-i}F(\mathcal{A})$ 必须在边界处为零，该条件等价于以下边界条件

$$F\left(\underset{\Leftarrow}{\mathcal{A}}^{+}\right)=-\frac{2\pi i}{\kappa k}\underset{\Leftarrow}{\mathcal{E}} \tag{60}$$

where the arrow denotes pullback to the boundary H. Condition (60) imposes the coupling between bulk and boundary degrees of freedom and, as will be discussed in section "Application: Black Hole Entropy," plays a major role in studying boundaries in the quantum theory such as the context of quantum supersymmetric black holes.

其中箭头表示拉回至边界 H 。条件 (60) 规定了体自由度与边界自由度之间的耦合，正如章节“应用: 黑洞熵”将要讨论的，它在研究量子理论中的边界问题 (例如量子超对称黑洞相关问题) 时起到关键作用

The Quantum Theory and SUSY Spin Nets

量子理论与超对称自旋网

Having discussed the geometric structure the Holst-MacDowell-Mansouri action in the chiral limit, following [51, 52, 76] , let us next turn to the quantum theory. By varying the bulk action in (58), one obtains

沿着 [51, 52, 76] 的思路，我们已经讨论了手征极限下霍尔斯特-麦克道威尔-曼苏里作用量的几何结构，接下来我们转向量子理论。对 (58) 中的体作用量变分可得

$$\delta S_{\text{bulk}}(\mathcal{A})=\frac{i}{\kappa}\int_{M}\left\langle D^{(\mathcal{A}^{+})}\delta\mathcal{A}^{+}\wedge\mathcal{E}\right\rangle=:\mathrm{d}\Theta+\frac{i}{\kappa}\int_{M}\left\langle\delta\mathcal{A}^{+}\wedge D^{(\mathcal{A}^{+})}\mathcal{E}\right\rangle \tag{61}$$

with pre-symplectic potential $\Theta(\delta)$ which induces the bulk pre-symplectic structure

其中预辛势为 $\Theta(\delta)$ ，它诱导出体预辛结构

$$\Omega_{\text{bulk}}(\delta_1,\delta_2)=\frac{2i}{\kappa}\int_{\Sigma}\left\langle\delta_{[1}\mathcal{A}^{+}\wedge\delta_{2]}\mathcal{E}\right\rangle=\frac{i}{\kappa}\int_{\Sigma}\mathrm{d}^3x\left(\delta_1\mathcal{A}_a^{+}\underline{A}_2\mathcal{E}_{\underline{A}}^{a}-\delta_2\mathcal{A}_a^{+}\underline{A}_1\mathcal{E}_{\underline{A}}^{a}\right)$$

(62)

Here,

此处,

$$\mathcal{E}_{\underline{A}}^{a}:=\frac{1}{2}\varepsilon^{abc}\mathcal{T}_{\underline{BA}}\mathcal{E}_{bc}^{\underline{B}} \tag{63}$$

where we have chosen a homogeneous basis $\left(T_{\underline{A}}\right)_{\underline{A}}$ of $\mathfrak{osp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ and set $\mathcal{T}_{\underline{AB}}:=\left\langle T_{\underline{A}},T_{\underline{B}}\right\rangle$. Hence, it follows that $\left(\mathcal{A}_a^{+\underline{A}},\mathcal{E}_{\underline{A}}^{a}\right)$ generate a graded symplectic phase space of canonical chiral SUGRA and satisfy the graded Poisson bracket

其中我们选取了 $\mathfrak{osp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ 的齐次基 $\left(T_{\underline{A}}\right)_{\underline{A}}$ 并设 $\mathcal{T}_{\underline{AB}}:=\left\langle T_{\underline{A}},T_{\underline{B}}\right\rangle$。由此可得 $\left(\mathcal{A}_a^{+\underline{A}},\mathcal{E}_{\underline{A}}^{a}\right)$ 生成了正则手征超引力的分次辛相空间，并满足分次泊松括号

$$\left\{\mathcal{E}^a_{\underline{A}}(x), \mathcal{A}^{+B}_b(y)\right\} = i\kappa\delta^a_b\delta^B_{\underline{A}}\delta^{(3)}(x,y) \tag{64}$$

Moreover, from (61), we can immediately read off the bulk super Gauß constraint

此外，从 (61) 我们可以直接写出体超高斯约束

$$\mathcal{G}[\alpha] := \frac{i}{\kappa}\int_{\Sigma}\left\langle\alpha\wedge D^{(\mathcal{A}^+)}\mathcal{E}\right\rangle = -\frac{i}{\kappa}\int_{\Sigma}\left\langle\mathcal{E}\wedge D^{(\mathcal{A}^+)}\alpha\right\rangle + \frac{i}{\kappa}\int_{\Delta}\langle\mathcal{E},\alpha\rangle \tag{65}$$

where α is some $\mathfrak{osp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ -valued smearing function and $\Delta := \sum\cap\partial M$. This constraint satisfies $\{\mathcal{G}[\alpha], \mathcal{G}[\beta]\} = \mathcal{G}[[\alpha,\beta]]$ and therefore generates local $\mathrm{OSp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ gauge transformations on the phase space. Hence, it follows that the canonical chiral theory has an intriguing structure which turns out to be in complete analogy the bosonic self-dual theory. This suggests to quantize the bulk theory adapting and generalizing tools of standard LQG. In the following, let us briefly outline the construction of the quantum theory following [51,76]. The mathematical details underlying this construction can be found in [52,55].

其中 α 是取值在 $\mathfrak{osp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ 的抹色函数，且 $\Delta := \sum\cap\partial M$ 。该约束满足 $\{\mathcal{G}[\alpha], \mathcal{G}[\beta]\} = \mathcal{G}[[\alpha,\beta]]$ ，因此在相空间上生成局部 $\mathrm{OSp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ 规范变换。由此可得，正则手征理论具有引人注目的结构，与玻色子自对偶理论完全类似。这提示我们可以适配并推广标准圈量子引力的方法来对体理论量子化。下面我们将遵循文献 [51,76] 简要概述量子理论的构造过程，该构造背后的数学细节可见文献 [52,55]。

In standard LQG, the bulk Hilbert space $\mathfrak{H}^{\mathrm{bulk}}_{\gamma}$ associated with a graph γ embedded in the spatial slices $\sum$ of the spacetime manifold $M = \mathbb{R}\times\sum$ is obtained by considering spin network states, a class of states invariant under local gauge transformations. They are constructed via the contraction of matrix coefficients of irreducible representations of the underlying gauge group. For this, it is crucial that the representations under consideration form a tensor category. Finite-dimensional irreducible representations of the orthosymplectic series $\mathrm{OSp}(\mathcal{N}\mid 2)$ for $\mathcal{N} = 1,2$ have been intensively studied (see, e.g.,[77-80]). For the case $\mathcal{N} = 1$, these representations form a subcategory closed under the tensor product. The corresponding spin network states have been studied, for instance, in [73,74]. For the case $\mathcal{N} = 2$, the subclass of typical representations form such a category (see [77]).

在标准圈量子引力中，嵌入时空流形 $M = \mathbb{R}\times\sum$ 空间切片 $\sum$ 的图 γ 对应的体希尔伯特空间 $\mathfrak{H}^{\mathrm{bulk}}_{\gamma}$ 由自旋网态构造得到，自旋网态是一类在局部规范变换下不变的态。它们通过收缩底层规范群不可约表示的矩阵系数构造得到。对此，关键在于所考虑的表示构成一个张量范畴。对应 $\mathcal{N} = 1,2$ 的正交辛序列 $\mathrm{OSp}(\mathcal{N}\mid 2)$ 的有限维不可约表示已有大量研究 (例如见文献 [77-80])。对于 $\mathcal{N} = 1$ 情形，这些表示构成了张量积下封闭的子范畴，对应的自旋网态研究可见例如文献 [73,74]。对于 $\mathcal{N} = 2$ 情形，典型表示的子类构成这样的范畴 (见文献 [77])。

We now describe the construction of the super spin network states for a suitable subclass $\mathcal{P}_{\mathrm{adm}}$ of irreducible representations (finite- or infinite-dimensional, possibly including those constructed in [51]) of $\mathrm{OSp}(\mathcal{N}\mid 2)$ with $\mathcal{N} = 1,2$. For any subset $\vec{\pi} := \{\pi_e\}_{e\in E(\gamma)} \subset \mathcal{P}_{\mathrm{adm}}$, we define the cylindrical function $T_{\gamma,\vec{\pi},\vec{m},\vec{n}}$ via

我们现在描述针对 OSp $(\mathcal{N}\mid 2)$ 满足 $\mathcal{N}=1,2$ 的不可约表示 (有限维或无限维，可能包含文献 [51] 中构造的那些表示) 的合适子类 $\mathcal{P}_{\text{adm}}$ ，构造超自旋网络态的过程。对于任意子集 $\vec{\pi}:=\{\pi_e\}_{e\in E(\gamma)}\subset\mathcal{P}_{\text{adm}}$ ，我们通过下式定义柱函数 $T_{\gamma,\vec{\pi},\vec{m},\vec{n}}$

$$T_{\gamma,\vec{\pi},\vec{m},\vec{n}}[\mathcal{A}^+]:=\prod_{e\in E(\gamma)}\pi_e(h_e[\mathcal{A}^+])^{m_e}{}_{n_e} \tag{66}$$

where, for any edge $e\in E(\gamma)$, $h_e[\mathcal{A}^+]$ denotes the super holonomy (parallel transport) of the connection $\mathcal{A}^+$ along e given by (It is interesting to note that the proper definition of the holonomy of a super connection requires the consideration of a more general class of supermanifolds called relative supermanifolds (see [55]) that depend on a certain parametrization. In this way, it follows that holonomies take values in a generalized Lie supergroup.) (see [52, 55] for more details)

其中，对于任意边 $e\in E(\gamma)$, $h_e[\mathcal{A}^+]$ 表示联络 $\mathcal{A}^+$ 沿 e 的超级和乐 (平行移动)，由下式给出 (值得注意的是，超联络和乐的恰当定义需要考虑一类更广义的超流形，称为相对超流形 (参见 [55])，这类超流形依赖于特定参数化。由此可得，和乐取值于广义李超群。)(更多细节参见 [52, 55])

$$h_e[\mathcal{A}^+]:=\mathcal{P}\exp\left(\int_e \mathrm{Ad}_{h_e[A]}\psi\right)\cdot h_e[A] \tag{67}$$

where we split $\mathcal{A}^+$ in the form $\mathcal{A}^+=A+\psi$ with A and ψ the underlying bosonic and fermionic subcomponent, respectively, with $A:=A^++\widehat{A}$ consisting of the self-dual Ashtekar connection A^+ and the $\mathfrak{so}(\mathcal{N})$ gauge field $\widehat{A}$. Furthermore, $(\pi_e)^{m_e}{}_{n_e}$ denote certain matrix coefficients of the representation $\pi_e\in\mathcal{P}_{\text{adm.}}$ In order to get a gauge-invariant state, at each vertex $v\in V(\gamma)$ of the graph γ, we have to contract (66) with an intertwiner I_v projecting onto the trivial representation at any vertex. As a result, the so-constructed state transforms trivially under local gauge transformations, i.e., it is annihilated by the quantum analog of the super Gauß constraint (65), and thus indeed forms a gauge-invariant state which we call a (gauge-invariant) super spin network state. We take these states as a basis of the state space of the bulk theory. We assume that an inner product can be found that turns this space into a super Hilbert space $\mathfrak{H}_\gamma^{\text{bulk}}$.

其中我们将 $\mathcal{A}^+$ 分解为 $\mathcal{A}^+=A+\psi$ 的形式，A 和 ψ 分别是对应的基础玻色子和费米子子分量，$A:=A^++\widehat{A}$ 包含自对偶阿西特卡联络 A^+ 和 $\mathfrak{so}(\mathcal{N})$ 规范场 $\widehat{A}$。此外，$(\pi_e)^{m_e}{}_{n_e}$ 表示表示 $\pi_e\in\mathcal{P}_{\text{adm.}}$ 的特定矩阵系数。为得到规范不变态，在图 γ 的每个顶点 $v\in V(\gamma)$ 处，我们需要将 (66) 与在每个顶点投影到平凡表示的缠结算子 I_v 缩并。由此构造的态在局部规范变换下平凡变换，即它被超高斯约束 (65) 的量子类比零化，因此确实构成了规范不变态，我们称之为 (规范不变) 超自旋网络态。我们将这些态作为体理论态空间的基底。我们假设存在内积可将该空间变为超希尔伯特空间 $\mathfrak{H}_\gamma^{\text{bulk}}$ 。

In this context, it is important to emphasize that one still needs to implement reality conditions as, a priori, one is dealing with complex variables. Following similar arguments as in the bosonic theory [81, 82], it seems highly suggestive that a solution of the reality conditions is given by selecting a specific subclass of representations of OSp $(\mathcal{N}\mid 2)_{\mathbb{C}}$ with respect to which geometric operators have particular nice and consistent properties. We will describe these kinds of representations in the following section. In section "Application: Black Hole Entropy," we will demonstrate that these representations indeed lead to consistent semi-classical results in the context of black hole computation.

在此背景下需要强调: 我们仍然需要落实实性条件，因为我们先验地处理的是复变量。遵循玻色理论 [81, 82] 中的类似论证，有很强的理由表明，通过选择 $\mathrm{OSp}\,(\mathcal{N}\mid 2)_{\mathbb{C}}$ 表示的一个特定子类即可得到实性条件的解，几何算符在该子类下具有尤其良好自洽的性质。我们将在下一节描述这类表示。在“应用: 黑洞熵”一节中，我们将证明这些表示在黑洞计算中确实能得到自洽的半经典结果。

The Super Area Operator

超面积算符

On the space of super spin networks, one can introduce a gauge-invariant operator in analogy to the area operator in standard LQG (see also section ”The Area Operator” in the context of higher dimensions). More precisely, since the super electric field

我们可以在超自旋网空间上，类比标准圈量子引力 (LQG) 中的面积算符引入一个规范不变算符 (另见高维语境下的“面积算符”小节)。更准确地说，由于超电场

$\mathcal{E}$ defines a $\mathfrak{osp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ -valued 2-form, for any oriented (semianalytic) surface S embedded in $\sum$, one can define the graded or super area quantity $\mathrm{gAr}\,(S)$ via

$\mathcal{E}$ 定义了一个取值为 $\mathfrak{osp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ 的 2-形式，因此对任意嵌入 $\sum$ 的定向 (半解析) 曲面 S ，我们可以通过下式定义分次或超面积量 $\mathrm{gAr}\,(S)$

$$\mathrm{gAr}\,(S)\;:=\;\sqrt{2}\int_S \parallel \mathcal{E} \parallel \tag{68}$$

where, in analogy to [83-85], the norm $\parallel \mathcal{E} \parallel$ on S is defined via $\parallel \mathcal{E} \parallel := \sqrt{\langle \mathcal{E}_S, \mathcal{E}_S\rangle}$ with $\mathcal{E}_S$ the unique $\mathfrak{osp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ -valued function on S such that $\iota_S^*\mathcal{E} = \mathcal{E}_S \mathrm{vol}_S$ with vol_S the volume form on S . For the special case $\mathcal{N} = 1$, the expression (68) coincides with the super area as considered in [74]. Here, the prefactor $\sqrt{2}$ has been chosen such that in the case of vanishing fermionic degrees of freedom, the super area reduces to the standard area of S in ordinary Riemannian geometry.

其中，类比文献 [83-85]，S 上的范数 $\parallel \mathcal{E} \parallel$ 通过 $\parallel \mathcal{E} \parallel := \sqrt{\langle \mathcal{E}_S, \mathcal{E}_S\rangle}$ 定义，$\mathcal{E}_S$ 是 S 上唯一的取值为 $\mathfrak{osp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ 的函数，满足 $\iota_S^*\mathcal{E} = \mathcal{E}_S \mathrm{vol}_S$ ，其中 vol_S 是 S 上的体积形式。对于特殊情况 $\mathcal{N} = 1$ ，表达式 (68) 与文献 [74] 中讨论的超面积一致。此处选择前因子 $\sqrt{2}$ ，是为了在费米子自由度为零时，超面积退化为普通黎曼几何中 S 的标准面积。

By definition, the quantity (68) solely depends on the super electric field which defines a phase space variable. Thus, we can implement it in the quantum theory by performing a similar regularization procedure in the bosonic theory (see [51, 52] for more details). As a result, for $\mathcal{N} = 1$, it follows, for instance, in the case that the surface S intersects the graph γ of a (gauge-invariant) super spin network state $T_{\gamma,\vec{\pi},\vec{m},\vec{n}}$ labeled by superspin quantum numbers $j \in \mathbb{C}$ corresponding to the principal series representations of OSp(1|2) as constructed in [51] in a single divalent vertex $v \in V\,(\gamma)$ that the action of super area operator is given by

根据定义，量 (68) 仅依赖作为相空间变量的超电场。因此我们可以通过在玻色理论中应用类似的正则化方案，将其引入量子理论 (更多细节见 [51, 52])。结果表明，对 $\mathcal{N}=1$，例如当曲面 S 与 (规范不变) 超自旋网态 $T_{\gamma,\vec{\pi},\vec{m},\vec{n}}$ 的图 γ 相交，且相交于单个二价顶点 $v\in V(\gamma)$ 时 (其中超自旋网态由超自旋量子数 $j\in\mathbb{C}$ 标记，对应文献 [51] 中构造的 OSp(1|2) 主系列表示)，超面积算符的作用可表示为

$$\widehat{\mathrm{gAr}}(S)\,T_{\gamma,\vec{\pi},\vec{m},\vec{n}} = -8\pi i l_p^2\sqrt{j\left(j+\frac{1}{2}\right)}T_{\gamma,\vec{\pi},\vec{m},\vec{n}} \tag{69}$$

with $j\in\mathbb{C}$ the superspin quantum number labeling the edge $e\in E(\gamma)$ intersecting the surface S. For $j\in\frac{N_0}{2}$, this coincides with the result of [74].

其中 $j\in\mathbb{C}$ 是标记相交于曲面 S 的边 $e\in E(\gamma)$ 的超自旋量子数。对于 $j\in\frac{N_0}{2}$，该结果与文献 [74] 的结论一致。

According to (69), the super area operator has complex eigenvalues which seems to be physically inconsistent. In fact, so far, we have not implemented the reality conditions in the quantum theory. Similar as in the context of the bosonic theory [81,82], one may then argue that solving the reality conditions amounts to selecting a specific subclass of representations with respect to which the spectrum of the super area operator becomes purely real. Interestingly, as shown in [51], it turns out that the principal series of $\mathrm{OSp}(1\mid 2)_{\mathbb{C}}$ indeed contains a subclass of irreducible representations that lead to a real spectrum of the super area operator. To be more precise, if one considers the series of representations labeled by superspin quantum numbers of the form

根据式 (69)，超面积算符具有复本征值，这在物理上似乎不自洽。事实上，我们至今尚未在量子理论中引入实性条件。和玻色理论的情况类似 [81,82]，我们可以认为，求解实性条件等价于选取特定的表示子类，使得超面积算符在该子类下的谱完全是实的。有趣的是，正如文献 [51] 所示，$\mathrm{OSp}(1\mid 2)_{\mathbb{C}}$ 的主系列表示中确实存在一个不可约表示子类，使得超面积算符的谱为实数。更准确地说，如果考虑如下由超自旋量子数标记的表示系列:

$$j=-\frac{1}{4}+is\text{ with }s\in\mathbb{R} \tag{70}$$

one finds that the action of the super area operator on these super spin network states takes the form

我们会发现，超面积算符作用在这些超自旋网态上的形式为

$$\widehat{\mathrm{gAr}}(S)\,T_{\gamma,\vec{\pi},\vec{m},\vec{n}} = 8\pi l_p^2\sqrt{s^2+\frac{1}{16}}T_{\gamma,\vec{\pi},\vec{m},\vec{n}} \tag{71}$$

and therefore it follows that super spin network states whose edges are labeled by j satisfying (70) are indeed eigenstates of the super area operator with real eigenvalues. Interestingly, this is in complete analogy to the bosonic theory [81].

因此可得: 若超自旋网态的边由满足式 (70) 的 j 标记，则这些态确实是超面积算符具有实本征值的本征态。有趣的是，这与玻色理论的结论完全一致 [81]。

Application: Black Hole Entropy

应用: 黑洞熵

In this final section, we briefly want to review the results of the articles [51,52,76] studying physical applications of the formalism as outlined above. More precisely, in [51, 76] , inner boundaries in chiral loop quantum supergravity have been considered. In particular, the possibility to associate an entropy to such boundaries has been discussed by studying microstates generated by super spin network states piercing the boundary. In the macroscopic limit, the resulting entropy formula remarkably turns out to be consistent with semi-classical computations. Let us also note that in [67] applications of the formalism in the context of minisuperspace models have been discussed. There, a complete solution of the reality conditions as well as a consistent implementation of the constraint algebra has been achieved. However, to keep it short, we will not review these results in this section and refer the interested reader to the original literature.

在最后这一节中，我们将简要综述文献 [51,52,76] 中对上述形式体系的物理应用研究结果。更具体地说，在 [51, 76] 中，研究了手征回路量子超引力中的内边界。具体而言，研究者通过分析穿透边界的超自旋网态产生的微观态，讨论了给这类边界赋予熵的可能性。宏观极限下，得到的熵公式显著符合半经典计算结果。此外需要指出，[67] 讨论了该形式体系在超空间模型中的应用，在该文中实现了实在条件的完全求解以及约束代数的自洽实现。但为简洁起见，本文不在本节综述这些结果，感兴趣的读者可查阅原始文献。

As described in section "The Quantum Theory and SUSY Spin Nets," the quantum excitations of the bulk degrees of freedom are represented by super spin network states associated with the gauge supergroup $\mathrm{OSp}(\mathcal{N} \mid 2)_{\mathbb{C}}$. On the other hand, according to Equations (58) and (59), the unique boundary theory is described in terms of a $\mathrm{OSp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ super Chern-Simons theory. Hence, for a given finite graph γ embedded in $\sum$, one may define the Hilbert space $\mathfrak{H}_{\text{full},\gamma}$ w.r.t. γ of the full theory as the tensor product

正如“量子理论与 SUSY 自旋网”一节所述，体自由度的量子激发由对应规范超群 $\mathrm{OSp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ 的超自旋网态表示。另一方面，根据式 (58) 和 (59)，唯一的边界理论可以用 $\mathrm{OSp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ 超陈-西蒙斯理论描述。因此，对于嵌入 $\sum$ 的给定有限图 γ ，全理论关于 γ 的希尔伯特空间 $\mathfrak{H}_{\text{full},\gamma}$ 可定义为张量积

$$\mathfrak{H}_{\gamma}^{\text{full}} = \mathfrak{H}_{\gamma}^{\text{bulk}} \otimes \mathfrak{H}_{\gamma}^{\text{bdy}} \tag{72}$$

with $\mathfrak{H}_{\gamma}^{\text{bulk}}$ the Hilbert space of the quantized bulk degrees of freedom as constructed in section "The Quantum Theory and SUSY Spin Nets" and $\mathfrak{H}_{\gamma}^{\text{bdy}}$ the Hilbert space corresponding to the quantized super Chern-Simons theory on the boundary.

其中 $\mathfrak{H}_{\gamma}^{\text{bulk}}$ 是“量子理论与 SUSY 自旋网”一节构造的量子化体自由度希尔伯特空间，$\mathfrak{H}_{\gamma}^{\text{bdy}}$ 对应边界上量子化超陈-西蒙斯理论的希尔伯特空间。

On this Hilbert space, one needs to implement the boundary condition (60). To this end, at each puncture $p \in \mathcal{P}_{\gamma} := \gamma \cap \Delta$, one chooses a disk $D_{\varepsilon}(p)$ on Δ around p with radius $\varepsilon > 0$ and sets

在该希尔伯特空间上，我们需要施加边界条件 (60)。为此，对每个刺点 $p \in \mathcal{P}_\gamma := \gamma \cap \Delta$，在 Δ 上围绕 p 取一个半径为 $\varepsilon > 0$ 的圆盘 $D_\varepsilon(p)$，并令

$$\mathcal{E}[\alpha](p) := \lim_{\varepsilon\to 0}\int_{D_\varepsilon(p)} \langle\alpha,\mathcal{E}\rangle,\; F[\alpha](p) := \lim_{\varepsilon\to 0}\int_{D_\varepsilon(p)} \langle\alpha, F(\mathcal{A}^+)\rangle \tag{73}$$

By definition, these quantities (or suitable functions thereof) can be promoted to well-defined operators in the quantum theory. Thus, (60) yields the additional constraint equation

根据定义，这些量 (或其合适的函数) 可以提升为量子理论中定义良好的算符。因此，式 (60) 给出额外的约束方程

$$\mathbb{1}\otimes \widehat{F}_{\underline{A}}(p) = -\frac{2\pi i}{\kappa k}\widehat{\mathcal{E}}_{\underline{A}}(p)\otimes\mathbb{1} \tag{74}$$

at each puncture $p \in \mathcal{P}_\gamma$, in analogy to the bosonic theory [86, 87] . The quantized super electric flux $\widehat{\mathcal{E}}_{\underline{A}}(p)$ acts in terms of right- resp. left-invariant vector fields (see [51, 52]). Hence, from (74), one concludes that the Hilbert space of the quantized boundary degrees of freedom corresponds to the Hilbert space of a quantized $\mathrm{OSp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ super Chern-Simons theory on Δ with punctures $\mathcal{P}_\gamma$ and complex Chern-Simons level k (see Fig. 1). According to Boltzmann, this suggests to associate an entropy to the boundary by defining it as the logarithm of the number of Chern-Simons degrees of freedom generated by the super spin network edges piercing the boundary. Unfortunately, the (super) Chern-Simons theory with complex and non-compact gauge group is not well known. Moreover, it is not clear how to deal with the fact that the Chern-Simons level is purely imaginary. Interestingly, similar issues also seem to arise in the context of boundary theories in string theory [88].

在每个刺点 $p \in \mathcal{P}_\gamma$ 处成立，这类似玻色子理论 [86, 87] 。量子化超电通量 $\widehat{\mathcal{E}}_{\underline{A}}(p)$ 通过右不变和左不变矢量场作用 (参见 [51, 52])。因此，从式 (74) 可以推得，量子化边界自由度的希尔伯特空间对应带刺点 $\mathcal{P}_\gamma$ 的 Δ 上量子化 $\mathrm{OSp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ 超陈-西蒙斯理论的希尔伯特空间，且陈-西蒙斯能级为复数值 k (参见图 1)。根据玻尔兹曼理论，我们可以给边界关联熵: 将其定义为穿透边界的超自旋网边产生的陈-西蒙斯自由度数的对数。遗憾的是，带复非紧致规范群的 (超) 陈-西蒙斯理论目前尚未被充分理解。此外，陈-西蒙斯能级为纯虚数这一问题也尚无处理思路。有意思的是，弦论的边界理论中似乎也出现了类似问题 [88]。

One may therefore adapt the strategy of [82] in the context of the purely bosonic theory to the supersymmetric setting by studying a specific compact real form of $\mathrm{OSp}(\mathcal{N}\mid 2)_{\mathbb{C}}$ and then performing an analytic continuation to the corresponding complex Lie supergroup. More precisely, restricting to the minimal supersymmetric case $\mathcal{N} = 1$ in what follows, one considers the Chern-Simons theory with compact gauge supergroup UOSp(1|2) and integer Chern-Simons level $k = -12\pi/\kappa\Lambda_{\cos}$ and punctures labeled by finite-dimensional irreducible representations $\vec{j}$ of UOSp(1|2) with $j \in \frac{\mathbb{N}_0}{2}$. One then computes the number $\mathcal{N}_k\left(\vec{j}\right)$ of Chern-Simons degrees of freedom given by the dimension of the superconformal blocks. Finally, one performs an analytic continuation by replacing $j \to j = -\frac{1}{4} + is$ for some $s \in \mathbb{R}$ for each $j \in \vec{j}$ as well as $k \to ik$ in $\mathcal{N}_k\left(\vec{j}\right)$. In order to simplify the discussion, let us assume that the boundary H is topologically of the form $\mathbb{R}\times\mathbb{S}^2$, that is, the 2-dimensional slices Δ_t are topologically equivalent to 2-spheres. Furthermore, we consider the limit $k \to \infty$ corresponding to a vanishing cosmological constant $\Lambda \to 0$. Under these assumptions, it follows that the number of microstates $\mathcal{N}_\infty\left(\vec{j}\right)$ is given by the number of UOSp(1|2) gauge-invariant states, i.e., it can be identified with the number of trivial subrepresentations contained in the tensor product representation $\otimes_j \pi_j$. As shown in [51], $\mathcal{N}_\infty$ can be expressed by the following integral formula:

因此，我们可以将纯玻色子理论中文献 [82] 的策略推广到超对称框架: 研究 $\mathrm{OSp}(\mathcal{N} \mid 2)_{\mathbb{C}}$ 的一个特定紧实实形式, 随后解析延拓到对应的复李超群。更准确地说, 下文将限制在极小超对称情形 $\mathcal{N} = 1$ ，我们研究带紧致规范超群 UOSp(1|2)、整数陈-西蒙斯能级 $k = -12\pi/\kappa\Lambda_{\cos}$ 、以及由 $j \in \frac{\mathbb{N}_0}{2}$ 标记的 UOSp(1|2) 有限维不可约表示 $\vec{j}$ 标记 puncture 的陈-西蒙斯理论。随后我们计算由超共形块维数给出的陈-西蒙斯自由度数目 $\mathcal{N}_k\left(\vec{j}\right)$ 。最后，我们对 $\mathcal{N}_k\left(\vec{j}\right)$ 中的 $k \to ik$ 以及每个 $j \in \vec{j}$ 用 $s \in \mathbb{R}$ 替换 $j \to j = -\frac{1}{4} + is$ ，完成解析延拓。为简化讨论，我们假设边界 H 拓扑上形如 $\mathbb{R} \times \mathbb{S}^2$ ，即二维切片 Δ_t 拓扑等价于二维球面。此外，我们考虑对应宇宙学常数 $\Lambda \to 0$ 趋于零的极限 $k \to \infty$ 。在这些假设下，可得微观状态数 $\mathcal{N}_\infty\left(\vec{j}\right)$ 等于 UOSp(1|2) 规范不变态的数目，即它可以等同于张量积表示 $\otimes_j \pi_j$ 中包含的平凡子表示数目。如文献 [51] 所示，$\mathcal{N}_\infty$ 可由下面积分公式表示:

$$\mathcal{N}_\infty\left(\{n_l, j_l\}\right) = \frac{1}{2\pi}\int_0^\pi \mathrm{d}\theta \sin^2(2\theta)\left[4 - n + \sum_{i=1}^p n_i d_{j_l} \frac{\tan\left(d_{j_i}\theta\right)}{\tan\theta}\right]\prod_{l=1}^p\left(\frac{\cos\left(d_{j_l}\theta\right)}{\cos\theta}\right)^{n_l}$$

(75)

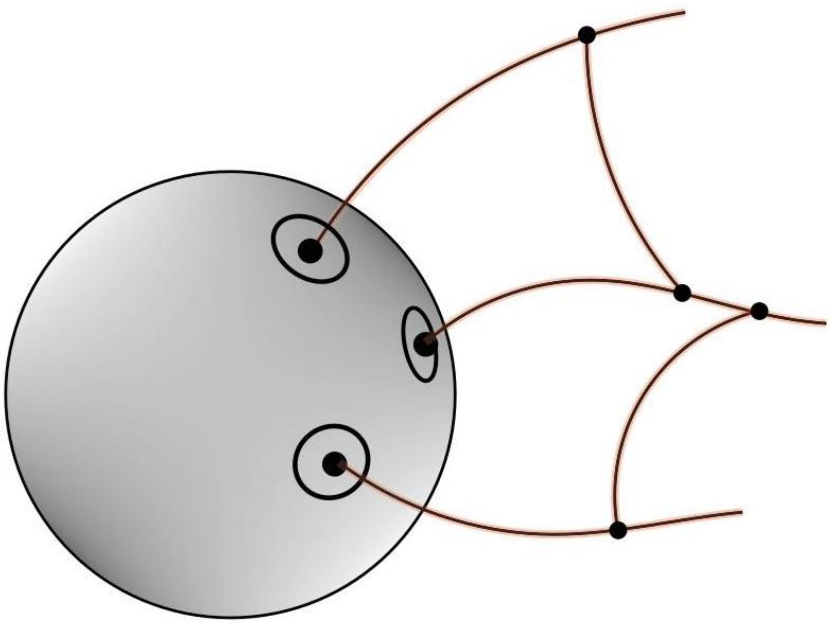

Fig. 1 Visualization of a supersymmetric black hole in chiral LQSG. The super spin network states induce nontrivial Chern-Simons degrees of freedom (black circles) at the intersection points (punctures) with the boundary which can account for black hole entropy. (Source [52])

图 1 手征 LQG 中超对称黑洞的可视化。超自旋网态在与边界的交点 (puncture) 处诱导出非平庸陈-西蒙斯自由度 (黑圈)，这些自由度可以解释黑洞熵。(来源 [52])

where $\vec{j}$ has been subdivided into $p \leq n := \left|\mathcal{P}_\gamma\right|$ subfamilies $\left(n_l, j_l\right), l = 1, \ldots, p$, corresponding to $0 < n_l \leq n$ punctures labeled by $j_l \in \vec{j}$. Moreover, $d_j := 4j + 1$ denotes the dimension of the finite-dimensional representation of $\mathrm{UOSp}(1 \mid 2)$ labeled by $j \in \frac{\mathbb{N}_0}{2}$. By performing an analytic continuation of

(75) following the algorithm as described above, one finds that the analytically continued state sum formula can be written as

> 其中 $\vec{j}$ 被划分为 $p \leq n := |\mathcal{P}_\gamma|$ 个子族 $(n_l, j_l), l = 1, \ldots, p$，对应由 $j_l \in \vec{j}$ 标记的 $0 < n_l \leq n$ 个 puncture。此外，$d_j := 4j + 1$ 表示 $\mathrm{UOSp}(1 \mid 2)$ 由 $j \in \frac{\mathbb{N}_0}{2}$ 标记的有限维表示的维数。按照上述算法对 (75) 做解析延推后，可以得到解析延拓后的状态和公式如下:

$$I_\infty = \frac{2}{\pi} \int_C \mathrm{d}z \mu(z) \left[1 - \frac{n}{4} - \sum_{i=1}^{p} n_i s_i \frac{\tan(s_i z)}{\tanh z}\right] \exp\left(\sum_{l=1}^{p} n_l \ln\left(\frac{\cos(s_l z)}{\cosh z}\right)\right)$$

(76)

where $\mu(z) := i\sinh^2(2z)$ and $\mathcal{C}$ is a contour from 0 to $i\pi$. The integral formula (76) can be evaluated approximately in the macroscopic limit corresponding to a large number of punctures $n_l \to \infty$ as well as large colors $s_l \to \infty \forall l = 1, \ldots, p$. In this limit and in the case of n indistinguishable punctures, one then finds that the entropy defined via $S := \ln(|\mathcal{I}_\infty| / n!)$ is given by

> 其中 $\mu(z) := i\sinh^2(2z)$ 和 $\mathcal{C}$ 是从 0 到 $i\pi$ 的围道。积分公式 (76) 可在对应大量刺点 $n_l \to \infty$ 与大色荷 $s_l \to \infty \forall l = 1, \ldots, p$ 的宏观极限下近似计算。在此极限且 n 个刺点不可区分的情况下，我们可得通过 $S := \ln(|\mathcal{I}_\infty| / n!)$ 定义的熵为

$$S = \ln\left(\frac{|\mathcal{I}_\infty|}{n!}\right) = \frac{a_H}{4 l_p^2} + O\left(\sqrt{a_H}\right) \tag{77}$$

with $a_H := 8\pi \sum_{l=1}^{p} n_l s_l$ the area of the boundary as measured w.r.t. the super area operator (69). Hence, in the macroscopic limit, it follows that the entropy associated with the boundary is indeed proportional to the area of the boundary in highest order with the correct prefactor of 1/4 as predicted by the semi-classical computations of Bekenstein and Hawking [89, 90]. Furthermore, the entropy acquires lower-order quantum corrections that can also be computed explicitly in the macroscopic limit (see [51] for more details).

> 其中 $a_H := 8\pi \sum_{l=1}^{p} n_l s_l$ 是超面积算符 (69) 测量得到的边界面积。因此在宏观极限下，边界关联的熵的最高阶项确实与边界面积成正比，且正比系数和贝肯斯坦与霍金半经典计算 [89,90] 预言的 1/4 一致。此外，熵获得了低阶量子修正，这些修正也可在宏观极限下显式计算 (更多细节见 [51])。

It is important to emphasize that Eq. (77) follows here directly from the analytically continued state sum formula (76). In particular, we did not have to make any choices or fix the Barbero-Immirzi parameter to specific values. Moreover, this confirms the results of [81, 82] in the bosonic theory and supports the hypothesis that, in the context of complex variables, the entropy can be derived via an analytic continuation starting from a compact real form of the complex gauge group. In the future, it would be very desirable to generalize these results to include supergravity theories with extended supersymmetry. In this respect, the pure $D = 4, \mathcal{N} = 4$ case would be of particular interest as it has been intensively studied in the string theory literature [91].

需要强调的是，式 (77) 在这里直接由解析延拓后的态和公式 (76) 得到。尤其我们不需要做任何选择，也不需要将巴贝罗-伊米里齐参数固定为特定数值。此外，这证实了玻色理论中 [81, 82] 的结果，也支持了如下假设: 在复变量框架下，可以从复规范群的紧致实形式出发，通过解析延推导出黑洞熵。未来非常希望将这些结果推广到包含扩展超对称的超引力理论中。在这方面，纯 $D = 4, \mathcal{N} = 4$ 的情况尤其值得关注，因为它已经在弦理论文献中被深入研究 [91]。

In any case, this example demonstrates that the techniques as discussed in this chapter can indeed be used for concrete physical applications leading to results that are consistent with semi-classical computations and which may provide a starting point for linking and testing ideas from LQG and string theory.

无论如何, 这个例子说明本章讨论的方法确实可用于具体物理应用, 得到的结果与半经典计算一致, 也可为连接和检验圈量子引力与弦理论的思想提供起点。

Modified Gravity

修正引力

Historically, GR serves as the simplest relativistic theory of gravity with correct Newtonian limit. It is worth pursuing all alternatives, which provide a high chance to uncover new physics. Recently, loop quantum gravity(LQG) has been generalized to the metric $f(\mathcal{R})$ theories [92,93], Brans-Dicke theory [94], and scalar-tensor theories (in both four-dimensional case [95] and higher-dimensional case [96]) and lowest-order projectable Hořava-Lifshitz gravity [97]. The fact that this background-independent quantization method can be successfully extended to those modified theories of gravity relies on the key observation that these gravity theories can also be reformulated into the connection dynamical formalism with a compact structure group. The purpose of this chapter is to review how to get the connection dynamics of these modified gravity theories and how to quantize these theories by the nonperturbative loop quantization procedure. We will use the scalar-tensor theories (STT) as the representative theories to illustrate our methodology. We only focus on the canonical quantization methods and omit the cosmological applications of loop quantum modified gravity which can be found in Refs. [98-100].

历史上，广义相对论 (GR) 是最简单的具有正确牛顿极限的相对论引力理论。探究所有替代理论是很有价值的, 这些理论有很高概率能揭示新物理。近年来, 圈量子引力 (LQG) 已被推广至度量 $f(\mathcal{R})$ 理论 [92,93]、布兰斯-迪克理论 [94]、标量-张量理论 (分为四维情形 [95] 与高维情形 [96]) 以及最低阶可投影霍拉瓦-李夫希茨引力 [97]。这种背景独立量子化方法能够成功推广到这些修正引力理论，关键在于我们发现这些引力理论也可以重新表述为具有紧结构群的联络动力学形式。本章的目的是综述如何得到这些修正引力理论的联络动力学, 以及如何通过非微扰圈量子化过程对这些理论进行量子化。我们将以标量-张量理论 (STT) 作为代表来说明我们的研究方法。我们仅关注正则量子化方法，省略了圈量子修正引力的宇宙学应用，相关内容可参见文献 [98-100]。

General Scheme

通用框架

In this section, we will first outline the general scheme of loop quantization for metric modified gravity theories [101]. Here, we are mainly focusing on four-dimensional metric theories of gravity and assume the modified gravity theory which is under consideration has a well-defined geometrical dynamics description. In this sense, the Hamiltonian formalism with 3-metric q_{ab} as one of the configuration variables is required. Moreover, the constraint algebra should be first class (or after solving some second-class constraints). Without loss of generality, we suppose the classical phase space of this theory consists of conjugate pairs $\left(q_{ab}, p^{ab}\right)$ and $\left(\phi_A, \pi^A\right)$ with ϕ_A being a scalar, vector, tensor, or spinor field. Then our quantization scheme has the following recipe.

在本节中，我们将首先概述度量修正引力理论的圈量子化通用框架 [101]。本文主要研究四维度量引力理论，并假设所讨论的修正引力理论具备明确的几何动力学描述。在此要求下，需要以三维度量 q_{ab} 作为构型变量之一建立哈密顿形式体系。此外，约束代数应为第一类 (或求解部分第二类约束后得到第一类)。不失一般性，我们假设该理论的经典相空间由共轭对 $\left(q_{ab}, p^{ab}\right)$ 和 $\left(\phi_A, \pi^A\right)$ 构成，其中 ϕ_A 为标量场、矢量场、张量场或旋量场。我们的量子化方案遵循以下步骤。

1, To obtain the corresponding connection dynamical formalism of modified gravity theories, we first introduce a quantity $\widetilde{K}_{ab}$ via

1. 为得到修正引力理论对应的联络动力学形式体系，我们首先通过下式引入物理量 $\widetilde{K}_{ab}$

$$\widetilde{K}_{ab} = \frac{16\pi G}{\sqrt{h}}\left(p_{ab} - \frac{1}{2}p q_{ab}\right). \tag{78}$$

Then we enlarge the phase space to obtain the triad formulation as

随后我们扩张相空间，得到三重域形式如下:

$$\left(q_{ab}, p^{ab}\right) \Rightarrow \left(E^a_i \equiv \sqrt{q} q_{ab} e^b_i, \widetilde{K}^i_a \equiv \widetilde{K}_{ab} e^b_i\right). \tag{79}$$

Now we make a canonical transformation to connection formulation as

现在我们对联络形式体系做正则变换，得到:

$$\left(E^a_j, \widetilde{K}^j_a\right) \Rightarrow \left(E^a_j, A^j_a \equiv \Gamma^j_a + \gamma \widetilde{K}^j_a\right), \tag{80}$$

and to recover the symmetric property of p_{ab} , we have to impose the condition $\widetilde{K}_{a[i}E^a_{j]} = 0$. This turns out to be the desired Gaussian constraint, $\mathcal{D}_a E^a_i \equiv \partial_a E^a_i + \varepsilon_{ijk} A^j_a E^a_k = 0$, which justified the internal structure group we adopted. By using these new variables, we can write all the constraints in terms of the new variables straightforwardly.

为恢复 p_{ab} 的对称性，我们需要施加条件 $\widetilde{K}_{a[i}E^a_{j]} = 0$ ，该条件即为所求的高斯约束 $\mathcal{D}_a E^a_i \equiv \partial_a E^a_i + \varepsilon_{ijk} A^j_a E^a_k = 0$ ，它证明了我们采用的内部结构群是合理的。利用这些新变量，我们可以直接将所有约束改写为新变量的形式。

2, For loop quantization, we represent the fields (ϕ_A, π^A) via polymer-like representation, together with the LQG representation for the holonomy-flux algebra for the gravitational part. Then the resulted kinematical Hilbert space is $\mathcal{H}_{\text{kin}} := \mathcal{H}_{\text{kin}}^{\text{gr}} \otimes \mathcal{H}_{\text{kin}}^{\phi}$. All the basic operators and geometrical operators could be promoted as the well-defined operators in this Hilbert space. As in the standard LQG, the Gaussian and diffeomorphism constraints can be solved. Then we would get the gauge-invariant $\mathcal{H}_G$ and diffeomorphism-covariant Hilbert spaces $\mathcal{H}_{\text{Diff}}$, respectively. In order to implement quantum dynamics, the Hamiltonian constraint operator can be constructed at least in $\mathcal{H}_G$ or even at $\mathcal{H}_{\text{Diff}}$, although it usually could not be well defined in $\mathcal{H}_{\text{Diff}}$.

2. 对于圈量子化，我们通过类聚合表示表示场 (ϕ_A, π^A)，并对引力部分的环绕-通量代数采用 LQG 表示，最终得到的运动学希尔伯特空间为 $\mathcal{H}_{\text{kin}} := \mathcal{H}_{\text{kin}}^{\text{gr}} \otimes \mathcal{H}_{\text{kin}}^{\phi}$。所有基本算符和几何算符都可以提升为该希尔伯特空间中良定义的算符。与标准 LQG 一样，高斯约束和微分同胚约束可以被求解，我们将分别得到规范不变的 $\mathcal{H}_G$ 和微分同胚协变希尔伯特空间 $\mathcal{H}_{\text{Diff}}$。为了实现量子动力学，至少可以在 $\mathcal{H}_G$ 甚至 $\mathcal{H}_{\text{Diff}}$ 上构造哈密顿约束算符，尽管它通常无法在 $\mathcal{H}_{\text{Diff}}$ 上得到良定义。

Due to the extreme complication, the quantum dynamics of LQG is still an open issue. Thus, in the following sections, we will take STT as an example to carry out the steps 1 and 2 in the above scheme.

由于问题极度复杂，LQG 的量子动力学目前仍是开放问题。因此，在接下来的章节中，我们将以 STT 为例，具体实施上述方案中的步骤 1 和步骤 2。

Hamiltonian Dynamics

哈密顿动力学

The most general action of STT reads

标量-张量理论 (STT) 最一般的作用量形式为

$$S(g) = \frac{1}{16\pi G} \int_{\Sigma} d^4x \sqrt{-g} \left[\phi \mathcal{R} - \frac{\omega(\phi)}{\phi} (\partial_\mu \phi) \partial^\mu \phi - 2V(\phi) \right] \tag{81}$$

where $\mathcal{R}$ denotes the scalar curvature of spacetime metric $g_{\mu\nu}$ and the coupling parameter $\omega(\phi)$ and potential $V(\phi)$ are the coupling parameter and potential, respectively. By doing 3+1 decomposition of the spacetime, the four-dimensional scalar curvature has the following decomposition:

其中 $\mathcal{R}$ 表示时空度规 $g_{\mu\nu}$ 的标量曲率，耦合参数 $\omega(\phi)$ 和势 $V(\phi)$ 分别为耦合参数和势。对时空进行 3+1 分解后，四维标量曲率满足如下分解形式:

$$\mathcal{R} = K_{ab}K^{ab} - K^2 + R + \frac{2}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} n^\mu K \right) - \frac{2}{N\sqrt{q}} \partial_a \left(\sqrt{q} q^{ab} \partial_b N \right) \tag{82}$$

where K_{ab} is the extrinsic curvature of the three-dimensional spatial metric q_{ab}, $K \equiv K_{ab}q^{ab}$, R denotes the scalar curvature on the spatial surface $\sum$, n^μ is the unit normal of $\sum$, and N represents the lapse function. By Legendre transformation, the momenta conjugate to the dynamical variables q_{ab} and ϕ are defined, respectively, as

其中 K_{ab} 是三维空间度规 q_{ab} 的外曲率，$K\equiv K_{ab}q^{ab}, R$ 表示空间曲面上的标量曲率，$\sum, n^{\mu}$ 是 $\sum$ 的单位法向量，N 表示时移函数。通过勒让德变换，动力学变量 q_{ab} 和 ϕ 对应的共轭动量分别定义为

$$p^{ab}=\frac{\partial\mathcal{L}}{\partial\dot{q}_{ab}}=\frac{\sqrt{q}}{16\pi G}\left[\phi\left(K^{ab}-Kq^{ab}\right)-\frac{h^{ab}}{N}\left(\dot{\phi}-N^{c}\partial_{c}\phi\right)\right], \tag{83}$$

$$\pi=\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=-\frac{\sqrt{q}}{8\pi G}\left(K-\frac{\omega\left(\phi\right)}{N\phi}\left(\dot{\phi}-N^{c}\partial_{c}\phi\right)\right), \tag{84}$$

with N^c being the shift vector. Combining of the trace of Eq. (83) with Eq. (84) gives us

其中 N^c 为移位矢量。结合式 (83) 的迹与式 (84) 可得

$$\left(3+2\omega\left(\phi\right)\right)\left(\dot{\phi}-N^{a}\partial_{a}\phi\right)=\frac{16\pi GN}{\sqrt{h}}\left(\phi\pi-p\right). \tag{85}$$

From Eq. (85), it is easy to see that one extra constraint $S=p-\phi\pi=0$ emerges when $\omega\left(\phi\right)=-\frac{3}{2}$. Hence, we divided the theories into two sector with $\omega\left(\phi\right)\neq-\frac{3}{2}$ and $\omega\left(\phi\right)=-\frac{3}{2}$.

从式 (85) 不难看出，当 $\omega\left(\phi\right)=-\frac{3}{2}$ 时会额外多出一个约束 $S=p-\phi\pi=0$ 。因此我们将该理论分为 $\omega\left(\phi\right)\neq-\frac{3}{2}$ 和 $\omega\left(\phi\right)=-\frac{3}{2}$ 两个分支。

Sector of $\omega\left(\phi\right)\neq-3/2$

$\omega\left(\phi\right)\neq-3/2$ 扇区

In the $\omega\left(\phi\right)\neq-3/2$ sector, the Hamiltonian of STT can be derived as a linear combination of constraints similar to GR as

在 $\omega\left(\phi\right)\neq-3/2$ 扇区中，可将标量-张量理论的哈密顿量推导为类似广义相对论的约束线性组合，形式如下

$$H_{\text{total}}=\int_{\Sigma}d^{3}x\left(N^{a}V_{a}+NH\right), \tag{86}$$

with the smeared diffeomorphism and Hamiltonian constraints being

其中弥散微分同胚约束与哈密顿量约束为

$$V\left(\vec{N}\right)=\int_{\Sigma}d^{3}xN^{a}V_{a}=\int_{\Sigma}d^{3}xN^{a}\left(-2D^{b}\left(p_{ab}\right)+\pi\partial_{a}\phi\right), \tag{87}$$

$$H\left(N\right)=\int_{\Sigma}d^{3}xNH \tag{88}$$

with

满足

$$H = \frac{16\pi G}{\sqrt{q}} \left(\frac{p_{ab}p^{ab} - \frac{1}{2}p^2}{\phi} + \frac{(p-\phi\pi)^2}{2\phi(3+2\omega)} \right) + \frac{\sqrt{q}}{16\pi G}(-\phi R$$

$$+\frac{\omega(\phi)}{\phi}(D_a\phi)D^a\phi + 2D_aD^a\phi + 2V(\phi)\Big) . \tag{89}$$

By using the symplectic structure

利用辛结构

$$\{q_{ab}(x), p^{cd}(y)\} = \delta_a^{(c}\delta_b^{d)}\delta^3(x,y),$$

$$\{\phi(x), \pi(y)\} = \delta^3(x,y), \tag{90}$$

straightforward calculation shows that the constraints (87) and (89) comprise a first-class system similar to GR as

直接计算表明，约束 (87) 与 (89) 构成类似广义相对论的第一类系统，形式如下

$$\{V(\vec{N}), V(\vec{N}')\} = V([\vec{N}, \vec{N}']),$$

$$\{H(M), V(\vec{N})\} = -H(\mathcal{L}_{\vec{N}}M),$$

$$\{H(N), H(M)\} = V(ND^aM - MD^aN). \tag{91}$$

To further obtain the connection dynamical formalism of the STT, following the general scheme, we introduce

为进一步得到标量-张量理论的联络动力学形式，我们遵循一般框架引入

$$\widetilde{K}^{ab} = \phi K^{ab} + \frac{q^{ab}}{2N}(\dot{\phi} - N^c\partial_c\phi) = \phi K^{ab} + \frac{q^{ab}}{(3+2\omega)\sqrt{q}}(\phi\pi - p). \tag{92}$$

Then we can construct new canonical pairs $\left(E_i^a \equiv \sqrt{h}e_i^a, \widetilde{K}_a^i \equiv \widetilde{K}_{ab}e_i^b\right)$, where e_i^a is the 3-triad such that $q_{ab}e_i^ae_j^b = \delta_{ij}$. Now the symplectic structure (90) implies the following non-zero Poisson brackets:

随后我们可以构造新的正则对 $\left(E_i^a \equiv \sqrt{h}e_i^a, \widetilde{K}_a^i \equiv \widetilde{K}_{ab}e_i^b\right.$ ，其中 e_i^a 是 3-标架，满足 $q_{ab}e_i^ae_j^b = \delta_{ij}$ 。此时辛结构 (90) 给出如下非零泊松括号:

$$\left\{\widetilde{K}_a^j(x), E_k^b(y)\right\} = 8\pi G\delta_a^b\delta_k^j\delta(x,y). \tag{93}$$

Other Poisson brackets are all vanished. Note that for the symmetric property $\widetilde{K}^{ab} = \widetilde{K}^{ba}$, we need to impose an additional constraint:

其余泊松括号均为零。注意到由于 $\widetilde{K}^{ab} = \widetilde{K}^{ba}$ 的对称性，我们需要额外施加一个约束:

$$G_{jk} \equiv \widetilde{K}_{a[j} E^a_{k]} = 0. \tag{94}$$

Now we can make a second canonical transformation via defining

现在我们可以通过定义进行第二次正则变换

$$A^i_a = \Gamma^i_a + \gamma \widetilde{K}^i_a, \tag{95}$$

where Γ^i_a is the spin connection and γ is a non-zero real number. It is easy to check that A^j_a goes back to the Ashtekar-Barbero connection [34] for $\phi = 1$. The nonvanishing Poisson bracket among the new variables reads

其中 Γ^i_a 是自旋联络，γ 是非零实数。不难验证，当 $\phi = 1$ 时，A^j_a 退化为阿西特卡-巴贝罗联络 [34]。新变量之间的非零泊松括号为

$$\left\{A^j_a(x), E^b_k(y)\right\} = 8\pi G\gamma \delta^b_a \delta^j_k \delta(x, y). \tag{96}$$

Combining Eq. (94) with the compatibility condition $\nabla_a E^a_i = \partial_a E^a_i + \varepsilon_{ijk}\Gamma^j_a E^{ak} = 0$ give us the standard Gaussian constraint

联立方程 (94) 与相容性条件 $\nabla_a E^a_i = \partial_a E^a_i + \varepsilon_{ijk}\Gamma^j_a E^{ak} = 0$，我们得到标准高斯约束

$$\mathcal{G}_i = \mathcal{D}_a E^a_i \equiv \partial_a E^a_i + \varepsilon_{ijk} A^j_a E^{ak} = 0, \tag{97}$$

which justifies A^i_a is indeed an $su(2)$ -connection. The diffeomorphism and Hamiltonian constraints in terms of new variables up to Gaussian constraint read, respectively, as

这证明 A^i_a 确实是 $su(2)$ 联络。在高斯约束下，用新变量表示的微分同胚约束与哈密顿量约束分别为

$$V_a = \frac{1}{8\pi G\gamma} F^i_{ab} E^b_i + \pi \partial_a \phi, \tag{98}$$

$$\begin{aligned} H = {} & \frac{\phi}{16\pi G}\left[F^j_{ab} - \left(\gamma^2 + \frac{1}{\phi^2}\right)\varepsilon_{jmn}\widetilde{K}^m_a \widetilde{K}^n_b\right]\frac{\varepsilon_{jkl}E^a_k E^b_l}{\sqrt{q}} \\ & + \frac{1}{(3 + 2\omega(\phi))}\left(\frac{\left(\widetilde{K}^i_a E^a_i\right)^2}{8\pi G\phi\sqrt{q}} + 2\frac{\left(\widetilde{K}^i_a E^a_i\right)\pi}{\sqrt{q}} + \frac{8\pi G\pi^2\phi}{\sqrt{q}}\right) \\ & + \frac{1}{8\pi G}\left[\frac{\omega(\phi)}{2\phi}\sqrt{q}(D_a\phi)D^a\phi + \sqrt{q}D_a D^a\phi + \sqrt{q}V(\phi)\right], \end{aligned} \tag{99}$$

where $F^i_{ab} \equiv 2\partial_{[a}A^i_{b]} + \varepsilon^i{}_{kl}A^k_a A^l_b$ represents the curvature of connection A^i_a . The total Hamiltonian thus can be expressed as a linear combination

其中 $F^i_{ab} \equiv 2\partial_{[a}A^i_{b]} + \varepsilon^i{}_{kl}A^k_aA^l_b$ 代表联络 A^i_a 的曲率。因此总哈密顿量可以表示为线性组合形式

$$H_{\text{total}} = \int_{\Sigma} \Lambda^i \mathcal{G}_i + N^a V_a + NH. \tag{100}$$

It is not hard to check that the smeared Gaussian constraint $\mathcal{G}(\Lambda) := \int_{\Sigma} d^3x \Lambda^i(x)\mathcal{G}_i(x)$ generates $SU(2)$ internal gauge transformations, while the smeared diffeomorphism constraint $\mathcal{V}\left(\vec{N}\right) := \int_{\Sigma} d^3x N^a\left(V_a - A^i_a\mathcal{G}_i\right)$ generates spatial diffeomorphism transformations. Together with the smeared Hamiltonian constraint $H(N) = \int_{\Sigma} d^3x NH$, the constraints algebra has the following form:

不难验证，弥散高斯约束 $\mathcal{G}(\Lambda) := \int_{\Sigma} d^3x \Lambda^i(x)\mathcal{G}_i(x)$ 生成 $SU(2)$ 内部规范变换，弥散微分同胚约束 $\mathcal{V}\left(\vec{N}\right) := \int_{\Sigma} d^3x N^a\left(V_a - A^i_a\mathcal{G}_i\right)$ 生成空间微分同胚变换。结合弥散哈密顿量约束 $H(N) = \int_{\Sigma} d^3x NH$，约束代数具有如下形式:

$$\{\mathcal{G}(\Lambda), \mathcal{G}(\Lambda')\} = 8\pi G \mathcal{G}([\Lambda, \Lambda']), \tag{101}$$

$$\left\{\mathcal{G}(\Lambda), \mathcal{V}\left(\vec{N}\right)\right\} = -\mathcal{G}\left(\mathcal{L}_{\vec{N}}\Lambda,\right), \tag{102}$$

$$\{\mathcal{G}(\Lambda), H(N)\} = 0, \tag{103}$$

$$\left\{\mathcal{V}\left(\vec{N}\right), \mathcal{V}\left(\vec{N}'\right)\right\} = \mathcal{V}\left(\left[\vec{N}, \vec{N}'\right]\right), \tag{104}$$

$$\left\{\mathcal{V}\left(\vec{N}\right), H(M)\right\} = H\left(\mathcal{L}_{\vec{N}}M\right), \tag{105}$$

$$\begin{aligned} \{H(N), H(M)\} &= \mathcal{V}\left(ND^aM - MD^aN\right) \\ &+ \mathcal{G}\left((N\partial_aM - M\partial_aN)\, q^{ab}A_b\right) \\ &- \frac{\left[E^aD_aN, E^bD_bM\right]^i}{8\pi Gq}\mathcal{G}_i \\ &- \gamma^2\frac{\left[E^aD_a(\phi N), E^bD_b(\phi M)\right]^i}{8\pi Gq}\mathcal{G}_i. \end{aligned} \tag{106}$$

The STT of gravity in the sector $\omega(\phi) \neq -3/2$ now have already been cast into the su(2)-connection dynamical formalism which lays the foundation for further loop quantization.

$\omega(\phi) \neq -3/2$ 扇区的引力标量-张量理论现已被改写为 su(2) 联络动力学形式，为进一步的圈量子化奠定了基础。

Sector of $\omega(\phi) = -3/2$

$\omega(\phi) = -3/2$ 扇区

Now we turn to the sector of $\omega(\phi) = -3/2$. Equation (85) suggests that there exists an extra primary constraint $S = 0$, which we call "conformal" constraint; the name conformal will became clear in the later on. Hence, the total Hamiltonian in this sector can be written as a linear combination

现在我们转向 $\omega(\phi) = -3/2$ 区段。式 (85) 表明存在一个额外的初级约束 $S = 0$，我们称之为共形约束；这个名称的含义后续会阐明。因此，该区段的总哈密顿量可以写成如下线性组合

$$H_{\text{total}} = \int_{\Sigma} d^3x \left(N^a V_a + NH + \lambda S\right), \tag{107}$$

where the smeared diffeomorphism constraint $V\left(\vec{N}\right)$ is the same as (87), while the Hamiltonian and conformal constraints read, respectively,

其中弥散微分同胚约束 $V\left(\vec{N}\right)$ 与式 (87) 一致，哈密顿约束和共形约束分别为

$$\begin{aligned}
H(N) &= \int_{\Sigma} d^3x NH \\
&= \int_{\Sigma} d^3x N \left[\frac{16\pi G}{\sqrt{q}} \left(\frac{p_{ab}p^{ab} - \frac{1}{2}p^2}{\phi}\right)\right. \\
&\left. + \frac{\sqrt{q}}{16\pi G} \left(-\phi R - \frac{3}{2\phi} (D_a\phi) D^a\phi + 2D_a D^a\phi + 2V(\phi)\right)\right], \\
S(\lambda) &= \int_{\Sigma} d^3x \lambda S = \int_{\Sigma} d^3x \lambda (p - \phi\pi).
\end{aligned} \tag{108}$$

With the help of symplectic structure (90), straightforward calculations show that

借助辛结构 (90)，直接计算可得

$$\{H(M), V(\vec{N})\} = -H\left(\mathcal{L}_{\vec{N}} M\right), \; \{S(\lambda), V(\vec{N})\} = -S\left(\mathcal{L}_{\vec{N}} \lambda\right), \tag{109}$$

$$\{H(N), H(M)\} = V\left(ND^aM - MD^aN\right) + S\left(\frac{D_a\phi}{\phi}\left(ND^aM - MD^aN\right)\right),$$

(110)

$$\{S(\lambda), H(M)\} = H\left(\frac{\lambda M}{2}\right) + \int_{\Sigma} N\lambda\sqrt{q}\left(-2V(\phi) + \phi V'(\phi)\right). \tag{111}$$

Equation (111) implies that a secondary constraint must to impose to insure to preserve the constraints S and H as

式 (111) 表明，必须施加一个次级约束，以保证约束 S 和 H 得以保持，即

$$-2V(\phi)+\phi V'(\phi)=0. \tag{112}$$

It is clear that this constraint is second class and hence one has to solve it. For the vacuum case where the solutions of Eq. (112) are strictly constrained into either

显然该约束是第二类约束，因此必须对其求解。对于真空情形，式 (112) 的解被严格限制为以下两种情形之一

$V(\phi)=0$ or $V(\phi)=C\phi^2$, with C being constant. For these two theories, the action (81) is invariant under the following conformal transformation:

$V(\phi)=0$ 或 $V(\phi)=C\phi^2$ ，其中 C 为常数。对于这两种理论，作用量 (81) 在下述共形变换下保持不变:

$$\tilde{g}_{\mu\nu}\to e^{\lambda}g_{\mu\nu},\ \widetilde{\phi}\to e^{-\lambda}\phi. \tag{113}$$

Thus, besides diffeomorphism invariance, those theories has an extra conformal symmetry. The resulted Hamiltonian formalism of the theory is comprised of a set of the constraints (V,H,S) . Moreover, these constraints form a first-class system; in particular, the transformations on the phase space generated by the conformal constraint are as follows:

因此，除微分同胚不变性外，这些理论还额外具有共形对称性。该理论导出的哈密顿形式主义由一组约束 (V,H,S) 构成。此外，这些约束构成第一类系统；具体来说，共形约束在相空间上生成的变换如下:

$$\{q_{ab},S(\lambda)\}=\lambda h_{ab},\ \left\{P^{ab},S(\lambda)\right\}=-\lambda P^{ab}, \tag{114}$$

$$\{\phi,S(\lambda)\}=-\lambda\phi,\ \{\pi,S(\lambda)\}=\lambda\pi. \tag{115}$$

Due to this extra conformal symmetry (108), the resulted physical degrees of freedom of this special kind of STT are equal to those of GR.

由于这个额外的共形对称性 (108)，这类特殊标量-张量理论的物理自由度与广义相对论的物理自由度相等。

By the canonical transformations and shift to the new variables (95), the resulted connection dynamics of STT can be obtained, and the total Hamiltonian can be expressed as

通过正则变换并转换到新变量 (95)，可以得到标量-张量理论的联络动力学，总哈密顿量可表示为

$$H_{\text{total}}=\int_{\Sigma}\Lambda^{i}\mathcal{G}_{i}+N^{a}V_{a}+NH+\lambda S, \tag{116}$$

where the Gaussian and diffeomorphism constraints are the same as in sector $\omega(\phi) \neq -3/2$, while the Hamiltonian and the conformal constraints read, respectively,

其中高斯约束与微分同胚约束和 $\omega(\phi) \neq -3/2$ 区段中的形式一致，哈密顿约束和共形约束分别为

$$H = \frac{\phi}{16\pi G}\left[F_{ab}^{j} - \left(\gamma^2 + \frac{1}{\phi^2}\right)\varepsilon_{jmn}\widetilde{K}_a^m\widetilde{K}_b^n\right]\frac{\varepsilon_{jkl}E_k^aE_l^b}{\sqrt{q}}$$

$$+\frac{1}{8\pi G}\left[-\frac{3}{4\phi}\sqrt{q}\left(D_a\phi\right)D^a\phi + \sqrt{q}D_aD^a\phi + \sqrt{q}V\left(\phi\right)\right], \tag{117}$$

$$S = \frac{1}{8\pi G}\widetilde{K}_a^iE_i^a - \pi\phi. \tag{118}$$

These constraints again form a first-class system [95].

这些约束再次构成第一类系统 [95]。

Loop Quantum Kinematics and Dynamics of Scalar-Tensor Theories

标量张量理论的圈量子运动学与动力学

With the connection dynamics in hand, the nonperturbative loop quantization procedure now is ready to extend to the STT. The kinematical structure of STT keeps the same form as that of standard LQG coupled with a polymer scalar field [92, 93] which means the kinematical Hilbert space STT is also a direct product of the Hilbert space of the gravitational part and that of the polymer scalar field, $\mathcal{H}_{\text{kin}} := \mathcal{H}_{\text{kin}}^{\text{gr}} \otimes \mathcal{H}_{\text{kin}}^{\text{sc}}$. This Hilbert space $\mathcal{H}_{\text{kin}}$ admits an orthonormal basis which is so-called spin scalar network basis over some graph $\alpha \cup X \subset \sum$ as

在得到联络动力学后，我们就可以将非微扰圈量子化 procedure 推广到标量张量理论 (STT)。STT 的运动学结构与耦合了聚合物标量场的标准圈量子引力 (LQG) 保持相同形式 [92, 93],这意味着 STT 的运动学希尔伯特空间同样是引力部分希尔伯特空间与聚合物标量场希尔伯特空间的直积, $\mathcal{H}_{\text{kin}} := \mathcal{H}_{\text{kin}}^{\text{gr}} \otimes \mathcal{H}_{\text{kin}}^{\text{sc}}$ 。该希尔伯特空间 $\mathcal{H}_{\text{kin}}$ 存在一组正交归一基，也就是定义在图 $\alpha \cup X \subset \sum$ 上的自旋标量网络基，形式如下

$$T_{\alpha,X}\left(A,\phi\right) \equiv T_\alpha\left(A\right) \otimes T_X\left(\phi\right), \tag{119}$$

where α and X consist of a finite number of curves and points in $\sum$, respectively. The fundamental quantum operators of STT are the holonomy operator $h_e(A) = \mathcal{P}\exp\int_e A_a$ defined by a connection along edges $e \subset \sum$ and densitized triad operator $E(S,f) := \int_S \varepsilon_{abc}E_i^a f^i$ which smeared over 2-surfaces, while for the scalar field part, the basic operators are the point holonomies $U_\lambda = \exp\left(i\lambda\phi\left(x\right)\right)$ and momenta $\pi(R) := \int_R d^3x\pi\left(x\right)$ smeared on three-dimensional regions. As the characteristic feature of LQG, the spatial geometric operators are the same as those in standard LQG. Moreover, it is also natural to promote the Gaussian constraint $\mathcal{G}(\Lambda)$ as a well-defined operator [46] by the standard LQG way. The kernel of Gaussian constraint operator gives the internal gauge-invariant Hilbert space $\mathcal{H}_G$. Similar to LQG, the diffeomorphism constraint can be solved by the group averaging technique, and the desired diffeomorphism-covariant Hilbert space $\mathcal{H}_{\text{Diff}}$ will be obtained [46,95]. The remaining task then is to implement the quantum Hamiltonian operators on this gauge-invariant and diffeomorphism-covariant Hilbert space.

其中 α 和 X 分别由 $\sum$ 中有限条曲线和点构成。STT 的基本量子算符包括: 沿边 $e \subset \sum$ 由联络定义的和乐算符 $h_e(A) = \mathcal{P}\exp\int_e A_a$ ，以及弥散在二维曲面上的密化三分量标架算符 $E(S,f) := \int_S \varepsilon_{abc} E^a_i f^i$ ；而对于标量场部分，基本算符是点和乐 $U_\lambda = \exp(i\lambda\phi(x))$ 和弥散在三维区域上的动量 $\pi(R) := \int_R d^3x\pi(x)$。作为 LQG 的特征性质，空间几何算符与标准 LQG 中的形式完全一致。此外，我们也可以通过标准 LQG 的方法自然地将高斯约束 $\mathcal{G}(\Lambda)$ 提升为良定义的算符 [46]。高斯约束算符的核给出内规范不变希尔伯特空间 $\mathcal{H}_G$ 。和 LQG 类似，微分同胚约束可以通过群平均技术求解，最终得到我们需要的微分同胚协变希尔伯特空间 $\mathcal{H}_{\text{Diff}}$ [46, 95]。剩余的任务便是在该规范不变且微分同胚协变的希尔伯特空间上实现量子哈密顿算符。

Sector of $\omega(\phi) \neq -3/2$

$\omega(\phi) \neq -3/2$ 扇区

In order to implement the Hamiltonian constraint (99) at the quantum level, let us first write Eq. (99) as $H(N) = \sum_{i=1}^{8} H_i$ in regular order. The complete treatment of the Hamiltonian constraint can be found in [95]. Here, we just take the term $H_6 = \int_\sum d^3xN\frac{\omega(\phi)}{2\phi}\sqrt{h}(D_a\phi)D^a\phi$ as an example to give the complete process. This term is somehow like the kinetic term of a Klein-Gordon field. To begin with, we first introduce the well-defined operators $\widehat{\phi}, \widehat{\phi^{-1}}$ [93]. This enables us to quantize that function $\omega(\phi)$ in the way of Taylor expansion. Moreover, by the point-splitting regularization techniques as in Refs. [93,102], we first triangulate the spatial surface $\sum$ in adaptation to some graph α underling a cylindrical function in $\mathcal{H}_{\text{kin}}$. Moreover, we re-express connections by basic variables such as holonomies or triads. The action of the corresponding regulated operator on a basis vector $T_{\alpha,X}$ reads

为了在量子层面实现哈密顿约束 (99)，我们首先将式 (99) 按正规序写为 $H(N) = \sum_{i=1}^{8} H_i$ 。哈密顿约束的完整处理可参见文献 [95]，此处仅以项 $H_6 = \int_\sum d^3xN\frac{\omega(\phi)}{2\phi}\sqrt{h}(D_a\phi)D^a\phi$ 为例给出完整推导过程。该项类似克莱因-戈登场的动能项。首先，我们引入定义良好的算符 $\widehat{\phi}, \widehat{\phi^{-1}}$ [93]，这使得我们可以通过泰勒展开对函数 $\omega(\phi)$ 进行量子化。此外，借助文献 [93,102] 中的点分裂正则化技术，我们首先对空间曲面 $\sum$ 做三角剖分，使其适配 $\mathcal{H}_{\text{kin}}$ 中柱函数对应的底图 α 。我们再用全纯或者标架这类基本变量重新表述联络。对应的正则化算符作用在基矢 $T_{\alpha,X}$ 上的结果为

$$
\begin{aligned}
\widehat{H}^\varepsilon_6 \cdot T_{\alpha,X} = \lim_{\varepsilon\to 0} \frac{2^{17}N(v)\,\widehat{\omega}(\phi)}{3^6\gamma^4(i\lambda_0)^2(i\hbar)^4 8\pi G}\widehat{\phi}^{-1}(v)\,\chi_\varepsilon(v-v') \\
\times \sum_{v\in\alpha(v)}\frac{1}{E(v)}\sum_{v(\Delta)=v}\varepsilon(s_L s_M s_N)\,\varepsilon^{LMN}\,\widehat{U}^{-1}_{\lambda_0}\left(\phi\left(s_{s_L(\Delta_v)}\right)\right) \\
\times\left[\widehat{U}_{\lambda_0}\left(\phi\left(t_{s_L(\Delta_v)}\right)\right) - \widehat{U}_{\lambda_0}\left(\phi\left(s_{s_L(\Delta_v)}\right)\right)\right] \\
\times \mathrm{Tr}\left(\tau_i \widehat{h}_{s_M(\Delta_v)}\left[\widehat{h}^{-1}_{s_M(\Delta_v)}, \left(\widehat{V}_{U^\varepsilon_v}\right)^{3/4}\right]\widehat{h}_{s_N(\Delta_v)}\left[\widehat{h}^{-1}_{s_N(\Delta_v)}, \left(\widehat{V}_{U^\varepsilon_v}\right)^{3/4}\right]\right)
\end{aligned}
$$

$$\times \sum_{v' \in \alpha(v)} \frac{1}{E(v')} \sum_{v(\Delta')=v'} \varepsilon(s_I s_J s_K) \varepsilon^{IJK} \widehat{U}_{\lambda_0}^{-1} \left(\phi\left(s_{s_I(\Delta_{v'})}\right)\right)$$

$$\times \left[\widehat{U}_{\lambda_0}\left(\phi\left(t_{s_I(\Delta_{v'})}\right)\right) - \widehat{U}_{\lambda_0}\left(\phi\left(s_{s_I(\Delta_{v'})}\right)\right)\right]$$

$$\times \operatorname{Tr}\left(\tau_i \widehat{h}_{s_J(\Delta_{v'})}\left[\widehat{h}^{-1}_{s_J(\Delta_{v'})}, \left(\widehat{V}_{U^{\varepsilon}_{v'}}\right)^{3/4}\right] \widehat{h}_{s_K(\Delta_{v'})}\left[\widehat{h}^{-1}_{s_K(\Delta_{v'})}, \left(\widehat{V}_{U^{\varepsilon}_{v'}}\right)^{3/4}\right]\right) \cdot T_{\alpha,X}.$$

(120)

The detailed meaning of notations in Eq. (120) can be found in [93]. It is obvious that the action of $\widehat{H}^{\varepsilon}_6$ on $T_{\alpha,X}$ will change the graph. Some vertices at $t(s_I(v)) = \varepsilon$ for edges $e_I(t)$ starting from each high-valent vertex of α are added. As a consequence, when $\varepsilon \to 0$, the family of operators $\widehat{H}^{\varepsilon}_6(N)$ is not weakly convergent. We must use the so-called uniform Rovelli-Smolin topology induced by diffeomorphism-invariant states Φ_{Diff} to define the limit operator as

> 式 (120) 中各记号的详细含义可参见文献 [93]。显然，$\widehat{H}^{\varepsilon}_6$ 作用在 $T_{\alpha,X}$ 上会改变图结构: 新增了 α 每个高顶点处出发的边 $e_I(t)$ 在 $t(s_I(v)) = \varepsilon$ 处对应的顶点。因此，当 $\varepsilon \to 0$ 时，算符族 $\widehat{H}^{\varepsilon}_6(N)$ 不弱收敛。我们必须利用微分同胚不变态 Φ_{Diff} 诱导的所谓均匀罗韦利-斯莫林拓扑来定义极限算符，即

$$\Phi_{\text{Diff}}\left(\widehat{H}_6 \cdot T_{\alpha,X}\right) = \lim_{\varepsilon \to 0}\left(\Phi_{\text{Diff}}\left|\widehat{H}^{\varepsilon}_6\right| T_{\alpha,X}\right\rangle. \tag{121}$$

The limit is independent of ε which means regulator ε can be removed and the action of the limit operator reads

> 该极限与 ε 无关，这意味着可以移除正则化因子 ε ，极限算符的作用为

$$\widehat{H}_6 \cdot T_{\alpha,X} = \sum_{v \in V(\alpha)} \frac{2^{17} N(v) \widehat{\omega}(\phi)}{3^6 \gamma^4 (i\lambda_0)^2 (i\hbar)^4 E^2(v) 8\pi G} \widehat{\phi}^{-1}(v)$$

$$\times \sum_{v(\Delta)=v(\Delta')=v} \varepsilon(s_L s_M s_N) \varepsilon^{LMN} \widehat{U}_{\lambda_0}^{-1}\left(\phi\left(s_{s_L(\Delta)}\right)\right)$$

$$\times \left[\widehat{U}_{\lambda_0}\left(\phi\left(t_{s_L(\Delta)}\right)\right) - \widehat{U}_{\lambda_0}\left(\phi\left(s_{s_L(\Delta)}\right)\right)\right]$$

$$\times \operatorname{Tr}\left(\tau_i \widehat{h}_{s_M(\Delta)}\left[\widehat{h}^{-1}_{s_M(\Delta)}, \left(\widehat{V}_v\right)^{3/4}\right] \widehat{h}_{s_N(\Delta)}\left[\widehat{h}^{-1}_{s_N(\Delta)}, \left(\widehat{V}_v\right)^{3/4}\right]\right)$$

$$\times \varepsilon(s_I s_J s_K) \varepsilon^{IJK} \widehat{U}_{\lambda_0}^{-1}\left(\phi\left(s_{s_I(\Delta')}\right)\right)$$

$$\times \left[\widehat{U}_{\lambda_0}\left(\phi\left(t_{s_I(\Delta')}\right)\right) - \widehat{U}_{\lambda_0}\left(\phi\left(s_{s_I(\Delta')}\right)\right)\right]$$

$$\times \operatorname{Tr}\left(\tau_i \widehat{h}_{s_J(\Delta')}\left[\widehat{h}^{-1}_{s_J(\Delta')}, \left(\widehat{V}_v\right)^{3/4}\right] \widehat{h}_{s_K(\Delta')}\left[\widehat{h}^{-1}_{s_K(\Delta')}, \left(\widehat{V}_v\right)^{3/4}\right]\right) \cdot T_{\alpha,X}.$$

(122)

The first two terms in the Hamiltonian constraint (99) are just the Euclidean and Lorentz term multiply with some function of ϕ . Since the actions of the polymer scalar field and gravitational part are separated, the actions of these two terms are rather simple. For the rest of the five terms, the detailed expression can be found in Ref. [95]. Therefore, the total Hamiltonian constraint in this sector has been fully quantized and promoted as a well-defined operator $\widehat{H}(N) = \sum_{i=1}^{8} \widehat{H}_i$ on kinematical Hilbert space $\mathcal{H}_{\text{kin}}$. The construction of $\widehat{H}(N)$ insures it is internal gauge invariant and diffeomorphism covariant. Hence, the Hamiltonian operator is well defined at least in the gauge-invariant Hilbert space $\mathcal{H}_G$. However, it is still difficult to define $\widehat{H}(N)$ directly on $\mathcal{H}_{\text{Diff}}$. Moreover, the constraint algebra of STT does not form a Lie algebra. This leads to possible quantum anomaly after quantization.

哈密顿约束 (99) 的前两项正是欧几里得项与洛伦兹项乘以 ϕ 的某个函数。由于聚合物标量场与引力部分的作用是分离的，因此这两项的作用形式相当简单。其余五项的具体表达式可参见文献 [95]。至此，该扇区的总哈密顿约束已完成量子化，成为运动学希尔伯特空间 $\mathcal{H}_{\text{kin}}$ 上定义良好的算符 $\widehat{H}(N) = \sum_{i=1}^{8} \widehat{H}_i$ 。$\widehat{H}(N)$ 的构造保证了它满足内部规范不变性与微分同胚协变性。因此，哈密顿算符至少在规范不变希尔伯特空间 $\mathcal{H}_G$ 上是定义良好的。但要直接在 $\mathcal{H}_{\text{Diff}}$ 上定义 $\widehat{H}(N)$ 仍然十分困难。此外，标量-张量理论的约束代数不构成李代数，这会导致量子化后可能出现量子反常。

Sector of $\omega(\phi) = -3/2$

$\omega(\phi) = -3/2$ 扇区

In the special case of $\omega(\phi) = -3/2$, we need to promote the smeared conformal constraint (118) to be a well-defined operator on kinematical Hilbert space. Note that both ϕ and $\pi(R)$ for polymer scalar field are already well-defined operators. First we note that the following identity:

在 $\omega(\phi) = -3/2$ 的特殊情形下，我们需要将涂抹后的共形约束 (118) 提升为运动学希尔伯特空间上一个良定义的算符。注意聚合物标量场的 ϕ 和 $\pi(R)$ 都已经是良定义的算符。首先我们给出以下恒等式:

$$\widetilde{K} \equiv \int_{\Sigma} d^3x \widetilde{K}_a^i E_i^a = \gamma^{-\frac{3}{2}} \{H^E(1), V\} \tag{123}$$

where the Euclidean term $H^E(1)$ is defined as

其中欧几里得项 $H^E(1)$ 定义为

$$H^E(1) = \frac{1}{16\pi G} \int_{\Sigma} d^3x F_{ab}^j \frac{\varepsilon_{jkl} E_k^a E_l^b}{\sqrt{h}}. \tag{124}$$

Both H^E and the volume V have been quantized in LQG. Then we can easily promote $S(\lambda)$ as a well-defined operator on a given basis vector $T_{\alpha,X} \in \mathcal{H}_{\text{kin}}$ as

LQG 中已经对 H^E 和体积 V 完成了量子化。因此我们可以直接将 $S(\lambda)$ 提升为给定基矢 $T_{\alpha,X} \in \mathcal{H}_{\text{kin}}$ 上的良定义算符，形如

$$\hat{S}(\lambda)\cdot T_{\alpha,X}=\left(\sum_{v\in V(\alpha)}\frac{\lambda(v)}{\gamma^{3/2}8\pi G(i\hbar)}\left[\hat{H}^{E}(1),\hat{V}_v\right]-\sum_{x\in X}\lambda(x)\hat{\phi}(x)\hat{\pi}(x)\right)\cdot T_{\alpha,X}.$$

(125)

It is clear that $\hat{S}(\lambda)$ is internal gauge invariant and diffeomorphism covariant. Moreover, it will also change a given graph. Hence, it is already well defined in gauge-invariant Hilbert space $\mathcal{H}_G$. Other terms in the Hamiltonian constraint operator in this sector are similar to that in the sector of $\omega(\phi)\neq -3/2$. The difference is that now ω takes the particular value of $\omega=-3/2$. We write Eq. (117) as $H(N)=\sum_{i=1}^{5}H_i$ in regular order. It is easy to check that the terms H_1,H_2,H_4,H_5 keep the same form as those in the last subsection, while the remaining term H_3 can be quantized as

> 不难看出 $\hat{S}(\lambda)$ 是内部规范不变且微分同胚协变的，此外它也会改变给定图。因此它已经在规范不变希尔伯特空间 $\mathcal{H}_G$ 中良定义。该扇区哈密顿约束算符的其他项与 $\omega(\phi)\neq -3/2$ 扇区的对应项类似，区别仅在于此处 ω 取 $\omega=-3/2$ 的特殊值。我们将式 (117) 按正则序写为 $H(N)=\sum_{i=1}^{5}H_i$ 。不难验证项 H_1,H_2,H_4,H_5 保持了上一小节给出的形式，剩余项 H_3 可量子化为

$$\begin{aligned}\hat{H}_3\cdot T_{\alpha,X}=&-\sum_{v\in V(\alpha)}\frac{2^{16}N(v)}{3^5\gamma^4(i\lambda_0)^2(i\hbar)^4 8\pi G E^2(v)}\hat{\phi}^{-1}(v)\\&\times\sum_{v(\Delta)=v(\Delta')=v}\varepsilon(s_Ls_Ms_N)\,\varepsilon^{LMN}\hat{U}^{-1}_{\lambda_0}\left(\phi\left(s_{s_L(\Delta)}\right)\right)\\&\times\left[\hat{U}_{\lambda_0}\left(\phi\left(t_{s_L(\Delta)}\right)\right)-\hat{U}_{\lambda_0}\left(\phi\left(s_{s_L(\Delta)}\right)\right)\right]\\&\times\mathrm{Tr}\left(\tau_i\hat{h}_{s_M(\Delta)}\left[\hat{h}^{-1}_{s_M(\Delta)},\left(\hat{V}_v\right)^{3/4}\right]\hat{h}_{s_N(\Delta)}\left[\hat{h}^{-1}_{s_N(\Delta)},\left(\hat{V}_v\right)^{3/4}\right]\right)\\&\times\varepsilon(s_Is_Js_K)\,\varepsilon^{IJK}\hat{U}^{-1}_{\lambda_0}\left(\phi\left(s_{s_I(\Delta')}\right)\right)\\&\times\left[\hat{U}_{\lambda_0}\left(\phi\left(t_{s_I(\Delta')}\right)\right)-\hat{U}_{\lambda_0}\left(\phi\left(s_{s_I(\Delta')}\right)\right)\right]\\&\times\mathrm{Tr}\left(\tau_i\hat{h}_{s_J(\Delta')}\right)\left[\hat{h}^{-1}_{s_J(\Delta')},\left(\hat{V}_v\right)^{3/4}\right]\hat{h}_{s_K(\Delta')}\left[\hat{h}^{-1}_{s_K(\Delta')},\left(\hat{V}_v\right)^{3/4}\right])\cdot T_{\alpha,X}.\end{aligned}$$

(126)

Now the total Hamiltonian constraint operator $\hat{H}(N)=\sum_{i=1}^{5}\hat{H}_i$ is also a well-defined operator in kinematical Hilbert space $\mathcal{H}_G$.

> 至此总哈密顿约束算符 $\hat{H}(N)=\sum_{i=1}^{5}\hat{H}_i$ 也是运动学希尔伯特空间 $\mathcal{H}_G$ 中的一个良定义算符。

Master Constraint and the Physical Hilbert Space

主约束与物理希尔伯特空间

The quantum dynamics of LQG is remaining an open question; in order to find a viable way to obtain the possible physical Hilbert space, Thiemann introduces the master constraint program into LQG [103]. One can form the master constraint from an anomalous set of constraints. Hence, in this section, we generalize the master constraint for the above quantum STT. Parallel to the above sections, we also discuss master constraint with different $\omega(\phi)$. In the sector of $\omega(\phi) \neq -3/2$, the master constraint of the STT is similar to that in GR case [95], while, for the case of $\omega(\phi) = -3/2$, due to the existence of the extra conformal constraint, the master constraint for this sector should be generalized to include the conformal constraint as [95]

LQG 的量子动力学仍是一个开放问题；为了找到可行方法得到可能的物理希尔伯特空间，Thiemann 将主约束方案引入 LQG [103]。我们可以从一组反常约束构造主约束。因此，本节我们将对上述量子 STT 推广主约束。与前文对应，我们也讨论不同 $\omega(\phi)$ 下的主约束。在 $\omega(\phi) \neq -3/2$ 扇区中，STT 的主约束与 GR 情形下的主约束类似 [95]，而在 $\omega(\phi) = -3/2$ 情形下，由于存在额外共形约束，该扇区的主约束需要推广，以将共形约束包含在内，形如 [95]

$$\mathcal{M} := \frac{1}{2}\int_{\Sigma} d^3x \frac{|H(x)|^2 + |S(x)|^2}{\sqrt{h}}, \tag{127}$$

where the expressions of Hamiltonian constraint $H(x)$ and the conformal constraint $S(x)$ are given by Eqs. (117) and (118), respectively. It is easy to see that

其中哈密顿约束 $H(x)$ 和共形约束 $S(x)$ 的表达式分别由式 (117) 和 (118) 给出。不难看出

$$\mathcal{M} = 0 \Leftrightarrow H(N) = 0 \ \text{ and } \ S(\lambda) = 0 \ \forall N(x), \lambda(x). \tag{128}$$

It is not hard to check that now the master constraint together with Gauß constraint and diffeomorphism constraint forms a Lie algebra. After some suitable regularization methods [95], the quantum master constraint operator $\widehat{\mathcal{M}}$ acts on diffeomorphism-invariant states as

不难验证，现在主约束与高斯约束、微分同胚约束共同构成李代数。经过一些合适的正则化方法 [95]，量子主约束算符 $\widehat{\mathcal{M}}$ 作用在微分同胚不变态上，结果为

$$\left(\widehat{\mathcal{M}}\Phi_{\text{Diff}}\right) T_{s,c} = \lim_{\mathcal{P}\to\sum, \varepsilon, \varepsilon' \to 0} \Phi_{\text{Diff}} \left[\frac{1}{2} \sum_{c\in\mathcal{P}} \left(\hat{H}^{\varepsilon}_{C} \left(\hat{H}^{\varepsilon'}_{C}\right)^{\dagger} + \hat{S}^{\varepsilon}_{C} \left(\hat{S}^{\varepsilon'}_{C}\right)^{\dagger} \right) T_{s,c} \right]. \tag{129}$$

The operator $\widehat{\mathcal{M}}$ is positive and symmetric in $\mathcal{H}_{\text{Diff}}$ and admits a unique selfadjoint Friedrichs extension [93, 102]. Hence, it is also possible to obtain the physical Hilbert space of the quantum STT in this special case by the direct integral decomposition of $\mathcal{H}_{\text{Diff}}$ with respect to the spectrum of $\widehat{\mathcal{M}}$.

算符 $\widehat{\mathcal{M}}$ 在 $\mathcal{H}_{\text{Diff}}$ 中是正定对称的，且存在唯一自伴的弗里德里希延拓 [93, 102]。因此，对于这个特殊情形，我们也可以通过对 $\mathcal{H}_{\text{Diff}}$ 关于 $\widehat{\mathcal{M}}$ 的谱做直接积分分解，得到量子 STT 的物理希尔伯特空间。

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